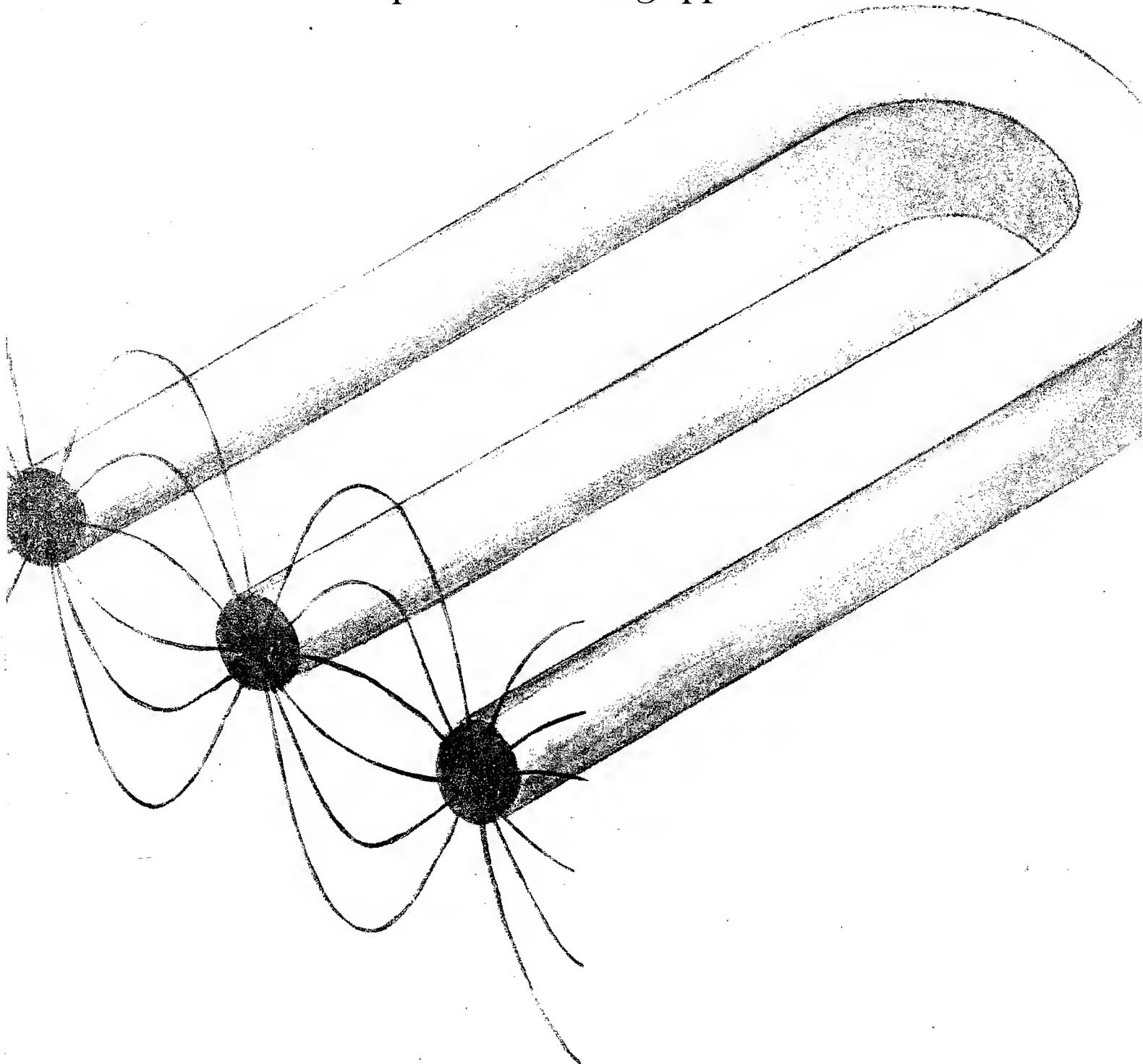


Markus Zahn

instructor's manual  
**ELECTROMAGNETIC  
FIELD THEORY**  
a problem solving approach





**Instructor's Manual**

**to accompany**

**ELECTROMAGNETIC FIELD THEORY:  
A PROBLEM SOLVING APPROACH**

**by**

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## PREFACE

To follow the problem solving approach of the text, there are a large number of exercise problems at the end of each chapter grouped together by topic and thus text section number. Using this manual as an aid together with the problem statements in the text, an instructor can easily choose those problems which reinforce the classroom materials.

The author would be appreciative of any corrections or other comments to this manual or to the text.

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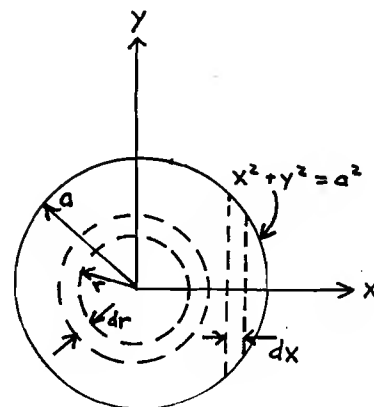
CHAPTER 1  
REVIEW OF VECTOR ANALYSIS

Section 1.1

$$1. \quad a) \quad \text{Area} = \int_{x=-a}^a \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dx dy$$

$$= 2 \int_{x=-a}^a \sqrt{a^2 - x^2} dx$$

$$= \left[ x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right]_{x=-a}^a = \pi a^2$$



$$b) \quad \text{Area} = \int_{r=0}^a \int_{\phi=0}^{2\pi} r dr d\phi$$

$$= 2\pi \left. \frac{r^2}{2} \right|_{r=0}^a = \pi a^2 \quad [\text{Easier than (a)}]$$

$$2. \quad a) \quad \text{Volume} = \int_{z=-R}^R \int_{y=-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} \int_{x=-\sqrt{R^2-y^2-z^2}}^{+\sqrt{R^2-y^2-z^2}} dx dy dz$$

$$= 2 \int_{z=-R}^R \int_{y=-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} \sqrt{R^2 - y^2 - z^2} dy dz$$

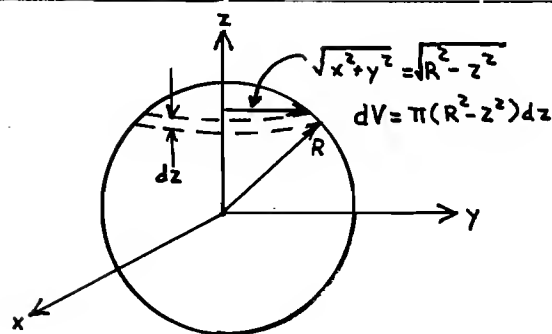
$$= \int_{z=-R}^R \left[ y \sqrt{R^2 - y^2 - z^2} + (R^2 - z^2) \sin^{-1} \frac{y}{\sqrt{R^2 - z^2}} \right] \bigg|_{y=-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} dz$$

$$= \pi \int_{z=-R}^R (R^2 - z^2) dz$$

$$= \pi \left( R^2 z - \frac{z^3}{3} \right) \bigg|_{z=-R}^R = \frac{4}{3} \pi R^3$$

# REVIEW OF VECTOR ANALYSIS

$$\begin{aligned}
 \text{b) Volume} &= \int_{\phi=0}^{2\pi} \int_{z=-R}^R \int_{r=0}^{\sqrt{R^2-z^2}} r \, dr \, d\phi \, dz \\
 &= \int_{z=-R}^R \pi(R^2 - z^2) \, dz \\
 &= \frac{4}{3} \pi R^3
 \end{aligned}$$



$$\begin{aligned}
 \text{c) Volume} &= \int_{r=0}^R \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
 &= \int_{r=0}^R -2\pi \cos\theta \Big|_{\theta=0}^{\pi} r^2 \, dr \\
 &= \int_{r=0}^R 4\pi r^2 \, dr \\
 &= \frac{4}{3} \pi r^3 \Big|_{r=0}^R = \frac{4}{3} \pi R^3 \quad [\text{Easier than (a) or (b)}]
 \end{aligned}$$

## Section 1.2

$$\begin{aligned}
 3. \quad \text{a) } \vec{A} + \vec{B} &= 6\vec{i}_x - 2\vec{i}_y - 6\vec{i}_z \\
 \vec{A} - \vec{B} &= 6\vec{i}_y + 4\vec{i}_z \\
 \vec{B} + \vec{C} &= 4\vec{i}_x - 5\vec{i}_y - 4\vec{i}_z \\
 \vec{B} - \vec{C} &= 2\vec{i}_x - 3\vec{i}_y - 6\vec{i}_z \\
 \vec{A} + \vec{C} &= 4\vec{i}_x + \vec{i}_y \\
 \vec{A} - \vec{C} &= 2\vec{i}_x + 3\vec{i}_y - 2\vec{i}_z
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \vec{A} \cdot \vec{B} &= 9 - 8 + 5 = 6 \\
 \vec{B} \cdot \vec{C} &= 3 + 4 - 5 = 2 \\
 \vec{A} \cdot \vec{C} &= 3 - 2 - 1 = 0
 \end{aligned}$$

# REVIEW OF VECTOR ANALYSIS

$$c) \quad \bar{A} \times \bar{B} = \det \begin{vmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 3 & 2 & -1 \\ 3 & -4 & -5 \end{vmatrix} = -14\bar{i}_x + 12\bar{i}_y - 18\bar{i}_z$$

$$\bar{B} \times \bar{C} = \det \begin{vmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 3 & -4 & -5 \\ 1 & -1 & +1 \end{vmatrix} = -9\bar{i}_x - 8\bar{i}_y + \bar{i}_z$$

$$\bar{A} \times \bar{C} = \det \begin{vmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \bar{i}_x - 4\bar{i}_y - 5\bar{i}_z$$

$$d) \quad \begin{aligned} (\bar{A} \times \bar{B}) \cdot \bar{C} &= (-14\bar{i}_x + 12\bar{i}_y - 18\bar{i}_z) \cdot (\bar{i}_x - \bar{i}_y + \bar{i}_z) = -44 \\ \bar{A} \cdot (\bar{B} \times \bar{C}) &= (3\bar{i}_x + 2\bar{i}_y - \bar{i}_z) \cdot (-9\bar{i}_x - 8\bar{i}_y + \bar{i}_z) = -44 \end{aligned} \quad \left. \vphantom{\begin{aligned} (\bar{A} \times \bar{B}) \cdot \bar{C} \\ \bar{A} \cdot (\bar{B} \times \bar{C}) \end{aligned}} \right\} \text{equal}$$

$$e) \quad \bar{A} \times (\bar{B} \times \bar{C}) = \det \begin{vmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 3 & 2 & -1 \\ -9 & -8 & 1 \end{vmatrix} = -6\bar{i}_x + 6\bar{i}_y - 6\bar{i}_z$$

$$\bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B}) = -6\bar{C} = -6(\bar{i}_x - \bar{i}_y + \bar{i}_z) \quad \text{equal}$$

$$f) \quad \cos \theta_{AC} = \frac{\bar{A} \cdot \bar{C}}{AC} = 0 \rightarrow \theta_{AC} = 90^\circ$$

$$\cos \theta_{B, \bar{A} \times \bar{C}} = \frac{\bar{B} \cdot (\bar{A} \times \bar{C})}{B |\bar{A} \times \bar{C}|} = \frac{44}{\sqrt{50(42)}} \approx .96 \rightarrow \theta_{B, \bar{A} \times \bar{C}} = 16.2^\circ$$

# REVIEW OF VECTOR ANALYSIS

$$4. \quad \bar{A} + \bar{B} = -\bar{i}_x + 5\bar{i}_y - 4\bar{i}_z$$

$$\bar{A} - \bar{B} = 3\bar{i}_x - \bar{i}_y - 2\bar{i}_z$$

$$\text{Add:} \quad 2\bar{A} = 2\bar{i}_x + 4\bar{i}_y - 6\bar{i}_z \rightarrow \bar{A} = \bar{i}_x + 2\bar{i}_y - 3\bar{i}_z$$

$$\text{Subtract:} \quad 2\bar{B} = -4\bar{i}_x + 6\bar{i}_y - 2\bar{i}_z \rightarrow \bar{B} = -2\bar{i}_x + 3\bar{i}_y - \bar{i}_z$$

$$5. \quad a) \quad \bar{B}_{||} = \alpha \bar{A}, \quad \bar{A} \cdot \bar{B} = AB_{||} = \alpha \bar{A} \cdot \bar{A} \rightarrow \alpha = \frac{\bar{A} \cdot \bar{B}}{\bar{A} \cdot \bar{A}}$$

$$b) \quad \bar{B}_{||} = \frac{\bar{A} \cdot \bar{B}}{\bar{A} \cdot \bar{A}} \bar{A} = \frac{(-12)}{6} (\bar{i}_x - 2\bar{i}_y + \bar{i}_z) = 2(-\bar{i}_x + 2\bar{i}_y - \bar{i}_z)$$

$$\bar{B}_{\perp} = \bar{B} - \bar{B}_{||} = 5\bar{i}_x + \bar{i}_y - 3\bar{i}_z$$

$$6. \quad \bar{A} \cdot \bar{B} = -36, \quad \bar{A} \cdot \bar{C} = 0 \quad (\bar{A} \text{ and } \bar{C} \text{ perpendicular}), \quad \bar{B} \cdot \bar{C} = 0 \quad (\bar{B} \text{ and } \bar{C} \text{ perpendicular})$$

$$\cos \theta_{AB} = \frac{\bar{A} \cdot \bar{B}}{AB} = \frac{-36}{\sqrt{24} \sqrt{54}} = -1 \rightarrow \theta_{A,B} = 180^\circ$$

$$7. \quad a) \quad \bar{A} \cdot \bar{B} = -75$$

$$b) \quad \bar{A} \times \bar{B} = \det \begin{vmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ 3 & 4 & 0 \\ 7 & -24 & 0 \end{vmatrix} = -100\bar{i}_z$$

$$c) \quad \cos \theta_{AB} = \frac{\bar{A} \cdot \bar{B}}{AB} = \frac{-75}{5(25)} = -.6 \rightarrow \theta_{AB} = 126.87^\circ$$

$$8. \quad \cos \alpha = \frac{\bar{A} \cdot \bar{i}_x}{A} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\cos \beta = \frac{\bar{A} \cdot \bar{i}_y}{A} = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$



# REVIEW OF VECTOR ANALYSIS

$$\cos \gamma = \frac{\vec{A} \cdot \vec{i}_z}{A} = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{A_x^2 + A_y^2 + A_z^2}{A_x^2 + A_y^2 + A_z^2} = 1$$

9. a)  $\vec{C} \cdot \vec{C} = (\vec{B} - \vec{A}) \cdot (\vec{B} - \vec{A}) = \vec{B} \cdot \vec{B} - 2\vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{A} \rightarrow C^2 = A^2 + B^2 - 2AB \cos \theta_c$

b)  $\vec{B} \times \vec{A} = (\vec{C} + \vec{A}) \times \vec{A} = \vec{C} \times \vec{A}$

$$|\vec{B} \times \vec{A}| = AB \sin \theta_c = AC \sin(\pi - \theta_b) = AC \sin \theta_b$$

$$\frac{\sin \theta_b}{B} = \frac{\sin \theta_c}{C}$$

$$\vec{B} \times \vec{C} = \vec{B} \times (\vec{B} - \vec{A}) = \vec{A} \times \vec{B}$$

$$|\vec{B} \times \vec{C}| = BC \sin \theta_a = |\vec{A} \times \vec{B}| = AB \sin \theta_c$$

$$\frac{\sin \theta_c}{C} = \frac{\sin \theta_b}{B} = \frac{\sin \theta_a}{A}$$

10. a)

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \det \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$(\vec{B} \times \vec{C}) \cdot \vec{A} = \det \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

$$(\vec{C} \times \vec{A}) \cdot \vec{B} = \det \begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix} = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

# REVIEW OF VECTOR ANALYSIS

- b) The vector  $\bar{A} \times \bar{B}$  is normal to the plane formed by the vectors  $\bar{A}$  and  $\bar{B}$  while its magnitude  $AB\sin\theta_{ab}$  gives the area of the parallelogram formed by  $\bar{A}$  and  $\bar{B}$ . The component of  $\bar{C}$  in the direction of  $\bar{A} \times \bar{B}$  defines the altitude of the parallelepiped formed by  $\bar{A}$ ,  $\bar{B}$ , and  $\bar{C}$ , so that the enclosed volume is

$$V = |(\bar{A} \times \bar{B}) \cdot \bar{C}|$$

c)

$$V = \bar{A} \times \bar{B} \cdot \bar{C} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 4$$

d)  $\bar{A} \times (\bar{B} \times \bar{C}) = (A_x \bar{i}_x + A_y \bar{i}_y + A_z \bar{i}_z) \times [(B_y C_z - B_z C_y) \bar{i}_x + (B_z C_x - B_x C_z) \bar{i}_y + (B_x C_y - B_y C_x) \bar{i}_z]$

$$= \bar{i}_x [A_y (B_z C_x - B_x C_z) - A_z (B_y C_x - B_x C_y)]$$

$$+ \bar{i}_y [-A_x (B_z C_y - B_y C_z) + A_z (B_x C_z - B_z C_y)]$$

$$+ \bar{i}_z [A_x (B_y C_z - B_z C_y) - A_y (B_x C_z - B_z C_y)]$$

$$= [B_x \bar{i}_x + B_y \bar{i}_y + B_z \bar{i}_z] [A_x C_x + A_y C_y + A_z C_z]$$

$$- [C_x \bar{i}_x + C_y \bar{i}_y + C_z \bar{i}_z] [A_x B_x + A_y B_y + A_z C_z]$$

$$= \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

11. a)  $\bar{A} = A \cos \theta \bar{i}_x + A \sin \theta \bar{i}_y$

$$\bar{B} = B \cos \phi \bar{i}_x - B \sin \phi \bar{i}_y$$

b)  $\bar{A} \cdot \bar{B} = AB \cos(\theta + \phi) = AB \cos \theta \cos \phi - AB \sin \theta \sin \phi$

$$\rightarrow \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

$$|\bar{A} \times \bar{B}| = AB \sin(\theta + \phi) = |-AB(\cos \theta \sin \phi + \sin \theta \cos \phi) \bar{i}_z|$$

$$\rightarrow \sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$$

## Section 1.3

12. a)  $\nabla f = \frac{\partial f}{\partial x} \bar{i}_x + \frac{\partial f}{\partial y} \bar{i}_y + \frac{\partial f}{\partial z} \bar{i}_z$

$$= (az + 3bx^2y) \bar{i}_x + bx^3 \bar{i}_y + ax \bar{i}_z$$

# REVIEW OF VECTOR ANALYSIS

$$\begin{aligned} \text{b) } \nabla f &= \frac{\partial f}{\partial r} \bar{i}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \bar{i}_\phi + \frac{\partial f}{\partial z} \bar{i}_z \\ &= \left(-\frac{a}{r^2} \sin \phi + bz^2 \cos 3\phi\right) \bar{i}_r + \left(\frac{a}{r^2} \cos \phi - 3bz^2 \sin 3\phi\right) \bar{i}_\phi + 2brz \cos 3\phi \bar{i}_z \end{aligned}$$

$$\begin{aligned} \text{c) } \nabla f &= \frac{\partial f}{\partial r} \bar{i}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \bar{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \bar{i}_\phi \\ &= \left(a \cos \theta - \frac{2b}{r^3} \sin \phi\right) \bar{i}_r - a \sin \theta \bar{i}_\theta + \frac{b}{r^3} \frac{\cos \phi}{\sin \theta} \bar{i}_\phi \end{aligned}$$

$$13. \quad f = r \sin \phi \rightarrow \nabla f = \sin \phi \bar{i}_r + \cos \phi \bar{i}_\phi = \bar{i}_y$$

$$\int_1 \nabla f \cdot d\bar{\ell} = \int_{-a}^{+a} dy = 2a$$

$$\int_2 \nabla f \cdot d\bar{\ell} = \int_{-\pi/2}^{+\pi/2} a \cos \phi d\phi = a \sin \phi \Big|_{-\pi/2}^{+\pi/2} = 2a$$

$$\int_3 \nabla f \cdot d\bar{\ell} = 2a$$

## Section 1.4

$$14. \quad \text{a) } \nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3$$

$$(\nabla \cdot \bar{A} = \frac{1}{r^2} \frac{\partial(r^3)}{\partial r} = 3 \text{ spherical coordinates})$$

$$\text{b) } \nabla \cdot \bar{A} = y^2 z^3 + 2xyz^3 + 3xy^2 z^2$$

$$\begin{aligned} \text{c) } \nabla \cdot \bar{A} &= \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\ &= 2 \cos \phi + \frac{\sin \phi}{r} \end{aligned}$$

# REVIEW OF VECTOR ANALYSIS

$$\begin{aligned} \text{d) } \nabla \cdot \bar{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \\ &= 4r \sin \theta \cos \phi + 2r \cos \theta \cos \phi - r \sin \phi \end{aligned}$$

$$15. \text{ a) } \int_V \nabla \cdot \bar{A} \, dV = \oint_S \bar{A} \cdot d\bar{S} \quad (\text{Divergence Theorem})$$

$$\text{Let } \bar{A} = \bar{i} f \rightarrow \nabla \cdot \bar{A} = (\bar{i} \cdot \nabla) f + f \nabla \cdot \bar{i}$$

$$\bar{i} \cdot \int_V \nabla f \, dV = \bar{i} \cdot \oint_S f \, d\bar{S} \rightarrow \int_V \nabla f \, dV = \oint_S f \, d\bar{S}$$

$$\text{b) } \text{Let } \bar{A} = \bar{i} \times \bar{F} \rightarrow \nabla \cdot \bar{A} = \nabla \cdot (\bar{i} \times \bar{F}) = \bar{F} \cdot (\nabla \times \bar{i}) - \bar{i} \cdot \nabla \times \bar{F} \quad (\text{See Prob. 24c})$$

$$\int_V \nabla \cdot \bar{A} \, dV = -\bar{i} \cdot \int_V \nabla \times \bar{F} \, dV$$

$$= \oint_S \bar{A} \cdot d\bar{S}$$

$$= \oint_S (\bar{i} \times \bar{F}) \cdot d\bar{S}$$

$$= \bar{i} \cdot \oint_S \bar{F} \times d\bar{S}$$

(See Prob. 10a)

$$\int_V \nabla \times \bar{F} \, dV = - \oint_S \bar{F} \times d\bar{S}$$

$$\text{c) } \text{Let } f = 1 \text{ in (a)} \rightarrow \nabla f = 0$$

$$\oint_S d\bar{S} = 0$$

$$\text{d) } \bar{n} = \bar{i}_r = \sin \theta \cos \phi \bar{i}_x + \sin \theta \sin \phi \bar{i}_y + \cos \theta \bar{i}_z$$

$$\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \bar{i}_r R^2 \sin \theta \, d\theta \, d\phi = R^2 [\bar{i}_x(0) + \bar{i}_y(0) - \frac{2\pi R^2}{4} \cos 2\theta \Big|_{\theta=0}^{\pi} \bar{i}_z] = 0$$

# REVIEW OF VECTOR ANALYSIS

$$\left. \begin{aligned} 16. \quad \nabla \cdot (f \nabla g) &= f \nabla^2 g + \nabla f \cdot \nabla g \\ \nabla \cdot (g \nabla f) &= g \nabla^2 f + \nabla f \cdot \nabla g \end{aligned} \right\} \text{ (See Prob. 24c)}$$

Subtract

$$\nabla \cdot (f \nabla g) - \nabla \cdot (g \nabla f) = f \nabla^2 g - g \nabla^2 f$$

$$\int_V \nabla \cdot [f \nabla g - g \nabla f] dV = \oint_S [f \nabla g - g \nabla f] \cdot d\vec{S} = \int_V [f \nabla^2 g - g \nabla^2 f] dV$$

$$17. \quad a) \quad d\vec{S}_1 = -dx dy \vec{i}_z$$

$$d\vec{S}_2 = -dx dz \vec{i}_y$$

$$d\vec{S}_3 = -dy dz \vec{i}_x$$

The equation of the plane surface 4 is of the form

$$x + Ay + Bz = C$$

where the constants A, B, and C are found by specifying three points in the plane.

$$(a, 0, 0)$$

$$(0, b, 0) \rightarrow f = x + \frac{a}{b} y + \frac{a}{c} z - a = 0$$

$$(0, 0, c)$$

The unit normal to the surface is found from  $\nabla f$  as

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{\vec{i}_x + \frac{a}{b} \vec{i}_y + \frac{a}{c} \vec{i}_z}{\sqrt{1 + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{c}\right)^2}}$$

The differential surface area is then

$$dS_4 = \frac{dx dy \vec{n}}{\cos \gamma}$$

where  $\gamma$  is the angle between surface 4 and the xy plane

$$\cos \gamma = \vec{n} \cdot \vec{i}_z = \frac{\frac{a}{c}}{\sqrt{1 + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{c}\right)^2}} \rightarrow d\vec{S}_4 = \frac{c}{a} dx dy \left[ \vec{i}_x + \frac{a}{b} \vec{i}_y + \frac{a}{c} \vec{i}_z \right]$$

$$b) \quad \vec{A} = r\vec{i}_r = x\vec{i}_x + y\vec{i}_y + z\vec{i}_z$$

$$\Phi = \oint_S \vec{A} \cdot d\vec{S}$$

$$= -\int_1 A_z(z=0) dx dy - \int_2 A_y(y=0) dx dz - \int_3 A_x(x=0) dy dz$$

$$+ \int_{x=0}^a \int_{y=0}^{b(1-\frac{x}{a})} \vec{A} \cdot d\vec{S}_4$$

$$z = -\frac{c}{a} \left( x + \frac{a}{b} y - a \right)$$

$$= \int_{x=0}^a \int_{y=0}^{b(1-\frac{x}{a})} \left[ x + \frac{a}{b} y + \frac{c}{a} z \right] \frac{c}{a} dx dy$$

$$= \int_{x=0}^a \int_{y=0}^{b(1-\frac{x}{a})} c dx dy$$

$$= \int_{x=0}^a cb \left[ 1 - \frac{x}{a} \right] dx$$

$$= \frac{abc}{2}$$

$$c) \quad \nabla \cdot \vec{A} = 3$$

$$\Phi = \int_V \nabla \cdot \vec{A} dV = 3 \underbrace{\frac{1}{3} \frac{abc}{2}}_{\text{volume of pyramid}} = \frac{abc}{2}$$

### Section 1.5

$$18. a) \quad \nabla \times \vec{A} = \vec{i}_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \vec{i}_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \vec{i}_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$= (x - y^2)\vec{i}_x - y\vec{i}_y - x^2\vec{i}_z$$

$$\begin{aligned}
 \text{b) } \nabla \times \bar{A} &= \left( \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \bar{i}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \bar{i}_\phi + \frac{1}{r} \left( \frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi} \right) \bar{i}_z \\
 &= -\sin\phi \bar{i}_r + \left( \frac{\sin\phi}{r} - \cos\phi \right) \bar{i}_\phi - \frac{z}{r^2} \cos\phi \bar{i}_z
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } \nabla \times \bar{A} &= \frac{1}{r \sin\theta} \left[ \frac{\partial}{\partial \theta} (A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi} \right] \bar{i}_r + \frac{1}{r} \left[ \frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (rA_\phi) \right] \bar{i}_\theta \\
 &\quad + \frac{1}{r} \left[ \frac{\partial}{\partial r} (rA_\theta) - \frac{\partial A_r}{\partial \theta} \right] \bar{i}_\phi \\
 &= -\frac{\cos\phi \cot\theta}{r^3} \bar{i}_r - r \sin\phi \bar{i}_\theta - \left[ \frac{\cos\theta \sin\phi}{r^3} + r \cos\theta \cos\phi \right] \bar{i}_\phi
 \end{aligned}$$

19. Let  $\bar{A} = f\bar{i}$ ,  $\nabla \times \bar{A} = \nabla f \times \bar{i} + f \nabla \times \bar{i}$  (See Prob. 24f)

$$\oint_L \bar{A} \cdot d\bar{\ell} = \bar{i} \cdot \oint_L f d\bar{\ell} = \int_S (\nabla \times \bar{A}) \cdot d\bar{S} = - \int_S (\bar{i} \times \nabla f) \cdot d\bar{S} = -\bar{i} \cdot \int_S \nabla f \times d\bar{S}$$

(See Prob. 10a)

$$\oint_L f d\bar{\ell} = - \int_S \nabla f \times d\bar{S}$$

20.  $\bar{A} = (x+a)(y+b)(z+c)\bar{i}_x$

$$\begin{aligned}
 \nabla \times \bar{A} &= -\frac{\partial A_x}{\partial y} \bar{i}_z + \frac{\partial A_x}{\partial z} \bar{i}_y \\
 &= (x+a)[-(z+c)\bar{i}_z + (y+b)\bar{i}_y]
 \end{aligned}$$

$$\begin{aligned}
 \oint_L \bar{A} \cdot d\bar{\ell} &= \int_{x=0}^{x_1} \int_{y=y_1}^{y_2} \int_{z=0}^0 A_x dx + \int_{x=x_1}^0 \int_{y=y_2}^0 \int_{z=0}^0 A_x dx \\
 &= \frac{(x+a)^2}{2} (y_1+b)c \Big|_{x=0}^{x_1} + \frac{(x+a)^2}{2} (y_2+b)c \Big|_{x=x_1}^0 \\
 &= \frac{c}{2} \{ (y_1+b)[(x_1+a)^2 - a^2] - (y_2+b)[(x_1+a)^2 - a^2] \} \\
 &= \frac{c}{2} [(x_1+a)^2 - a^2][y_1 - y_2]
 \end{aligned}$$

$$\int \nabla \times \bar{A} \cdot d\bar{S} = \iint (\nabla \times \bar{A})_z dx dy$$

flat surface  
at  $z=0$

$$= \int_{x=0}^{x_1} \int_{y=y_1}^{y_2} -c(x+a) dx dy = -\frac{c}{2} (y_2 - y_1) [(x_1 + a)^2 - a^2]$$

rectangular  
cylinder

$$\begin{aligned} \int \nabla \times \bar{A} \cdot d\bar{S} &= \int_{y=y_2 \text{ surface}} (\nabla \times \bar{A})_y dx dz - \int_{y=y_1 \text{ surface}} (\nabla \times \bar{A})_y dx dy + \int_{x=x_1 \text{ surface}} (\nabla \times \bar{A})_x dy dz \\ &\quad - \int_{x=0 \text{ surface}} (\nabla \times \bar{A})_x dy dz + \int_{z=z_1 \text{ surface}} (\nabla \times \bar{A})_z dx dy \\ &= \int_0^{z_1} \int_0^{x_1} (x+a)(y_2+b) dx dz - \int_0^{z_1} \int_0^{x_1} (x+a)(y_1+b) dx dz \\ &\quad - \int_0^{x_1} \int_0^{y_2} (z_1+c)(x+a) dx dy \\ &= (y_2+b) \frac{z_1}{2} [(x_1+a)^2 - a^2] - (y_1+b) \frac{z_1}{2} [(x_1+a)^2 - a^2] \\ &\quad - \frac{(z_1+c)(y_2-y_1)}{2} [(x_1+a)^2 - a^2] \\ &= -\frac{c}{2} [(x_1+a)^2 - a^2] [y_2 - y_1] \end{aligned}$$

$$21. \quad f = \frac{x^2 \ln y}{y}$$

$$\frac{\partial f}{\partial x} = \frac{2x \ln y}{y}; \quad \frac{\partial f}{\partial y} = \frac{x^2}{y^2} [1 - \ln y]$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{2x}{y^2} [1 - \ln y]; \quad \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{2x}{y^2} [1 - \ln y]$$

$$\begin{aligned} 22. \quad \nabla \cdot \bar{i}_x &= \nabla \cdot \bar{i}_y = \nabla \cdot \bar{i}_z = 0 \\ \nabla \times \bar{i}_x &= \nabla \times \bar{i}_y = \nabla \times \bar{i}_z = 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \nabla \cdot \bar{i}_x &= \nabla \cdot \bar{i}_y = \nabla \cdot \bar{i}_z = 0 \\ \nabla \times \bar{i}_x &= \nabla \times \bar{i}_y = \nabla \times \bar{i}_z = 0 \end{aligned}} \right\} \text{Cartesian}$$



# REVIEW OF VECTOR ANALYSIS

$$\begin{aligned}
 \nabla \cdot \bar{i}_r &= \frac{1}{r} \frac{\partial r}{\partial r} = \frac{1}{r}; \quad \nabla \cdot \bar{i}_\phi = 0; \quad \nabla \cdot \bar{i}_z = 0 \\
 \nabla \times \bar{i}_r &= 0; \quad \nabla \times \bar{i}_\phi = \frac{1}{r} \frac{\partial r}{\partial r} \bar{i}_z = \frac{1}{r} \bar{i}_z; \quad \nabla \times \bar{i}_z = 0 \\
 \nabla \cdot \bar{i}_r &= \frac{1}{r^2} \frac{\partial r^2}{\partial r} = \frac{2}{r}; \quad \nabla \cdot \bar{i}_\theta = \frac{1}{r \sin \theta} \frac{\partial \sin \theta}{\partial \theta} = \frac{\cot \theta}{r}; \quad \nabla \cdot \bar{i}_\phi = 0 \\
 \nabla \times \bar{i}_r &= 0; \quad \nabla \times \bar{i}_\theta = \frac{1}{r} \frac{\partial r}{\partial r} \bar{i}_\phi = \frac{1}{r} \bar{i}_\phi; \\
 \nabla \times \bar{i}_\phi &= \frac{1}{r \sin \theta} \frac{\partial \sin \theta}{\partial \theta} \bar{i}_r - \frac{1}{r} \frac{\partial r}{\partial r} \bar{i}_\theta = \frac{\cot \theta}{r} \bar{i}_r - \frac{1}{r} \bar{i}_\theta
 \end{aligned}$$

} Cylindrical

} Spherical

23.	<u>Cartesian</u>	<u>Cylindrical</u>	<u>Spherical</u>
a)	$h_x = 1$	$h_r = 1$	$h_r = 1$
	$h_y = 1$	$h_\phi = r$	$h_\theta = r$
	$h_z = 1$	$h_z = 1$	$h_\phi = r \sin \theta$

b)  $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial w} dw$

$$= \nabla f \cdot d\bar{l}$$

$$= \nabla f \cdot [h_u d\bar{i}_u + h_v d\bar{i}_v + h_w d\bar{i}_w]$$

$$(\nabla f)_u = \frac{1}{h_u} \frac{\partial f}{\partial u}; \quad (\nabla f)_v = \frac{1}{h_v} \frac{\partial f}{\partial v}; \quad (\nabla f)_w = \frac{1}{h_w} \frac{\partial f}{\partial w}$$

$$\nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} \bar{i}_u + \frac{1}{h_v} \frac{\partial f}{\partial v} \bar{i}_v + \frac{1}{h_w} \frac{\partial f}{\partial w} \bar{i}_w$$

c)  $dS_u = h_v h_w dv dw; \quad dS_v = h_u h_w du dw; \quad dS_w = h_u h_v du dv$

$$dV = h_u h_v h_w du dv dw$$

## d) Divergence

$$\begin{aligned}
 \phi &= \oint_S \bar{A} \cdot d\bar{S} = \int_1 A_u h_v h_w dv dw - \int_1' A_u h_v h_w dv dw \\
 &\quad u \quad u-\Delta u \\
 &\quad + \int_2 A_v h_u h_w du dw - \int_2' A_v h_u h_w du dw \\
 &\quad v+\Delta v \quad v \\
 &\quad + \int_3 A_w h_u h_v du dv - \int_3' A_w h_u h_v du dv \\
 &\quad w+\Delta w \quad w
 \end{aligned}$$

# REVIEW OF VECTOR ANALYSIS

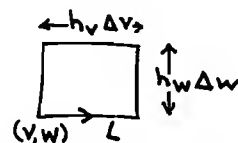
$$= \left\{ \frac{A_{u v w} \Big|_u - A_{u v w} \Big|_{u-\Delta u}}{\Delta u} + \frac{A_{v u w} \Big|_{v+\Delta v} - A_{v u w} \Big|_v}{\Delta v} \right. \\ \left. + \frac{A_{w u v} \Big|_{w+\Delta w} - A_{w u v} \Big|_w}{\Delta w} \right\} \Delta u \Delta v \Delta w$$

$$\nabla \cdot \bar{A} = \lim_{\substack{\Delta u \rightarrow 0 \\ \Delta v \rightarrow 0 \\ \Delta w \rightarrow 0}} \frac{\oint_S \bar{A} \cdot d\bar{S}}{\Delta V} = \frac{\oint_S \bar{A} \cdot d\bar{S}}{h_u h_v h_w \Delta u \Delta v \Delta w}$$

$$= \frac{1}{h_u h_v h_w} \left[ \frac{\partial (h_{v w} A_u)}{\partial u} + \frac{\partial (h_{u w} A_v)}{\partial v} + \frac{\partial (h_{u v} A_w)}{\partial w} \right]$$

Curl

$$(\nabla \times \bar{A})_u = \lim_{\substack{\Delta v \rightarrow 0 \\ \Delta w \rightarrow 0}} \frac{\oint_L \bar{A} \cdot d\bar{\ell}}{h_v h_w \Delta v \Delta w}$$



$$\oint_L \bar{A} \cdot d\bar{\ell} = \left[ A_{v v} h_v \Delta v \Big|_w - A_{v v} h_v \Delta v \Big|_{w+\Delta w} \right] + \left[ A_{w w} h_w \Delta w \Big|_{v+\Delta v} - A_{w w} h_w \Delta w \Big|_v \right]$$

$$(\nabla \times \bar{A})_u = \lim_{\substack{\Delta v \rightarrow 0 \\ \Delta w \rightarrow 0}} \frac{1}{h_v h_w} \left\{ \frac{[A_{v v} h_v \Big|_w - A_{v v} h_v \Big|_{w+\Delta w}]}{\Delta w} + \frac{[A_{w w} h_w \Big|_{v+\Delta v} - A_{w w} h_w \Big|_v]}{\Delta v} \right\} \\ = \frac{1}{h_v h_w} \left[ \frac{\partial (h_{w w} A_v)}{\partial v} - \frac{\partial (h_{v v} A_w)}{\partial w} \right]$$

Similarly

$$(\nabla \times \bar{A})_v = \frac{1}{h_u h_w} \left[ \frac{\partial (h_{u u} A_w)}{\partial w} - \frac{\partial (h_{w w} A_u)}{\partial u} \right]$$

$$(\nabla \times \bar{A})_w = \frac{1}{h_u h_v} \left[ \frac{\partial (h_{v v} A_u)}{\partial u} - \frac{\partial (h_{u u} A_v)}{\partial v} \right]$$

$$e) \nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_u h_v h_w} \left[ \frac{\partial}{\partial u} \left( \frac{h_{v w}}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{h_{u w}}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left( \frac{h_{u v}}{h_w} \frac{\partial f}{\partial w} \right) \right]$$

# REVIEW OF VECTOR ANALYSIS

$$\begin{aligned}
 25. \quad a) \quad \vec{r}_{QP} &= (x_2 - x_1)\vec{i}_x + (y_2 - y_1)\vec{i}_y + (z_2 - z_1)\vec{i}_z \\
 r_{QP} &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \\
 &= [1^2 + (-5)^2 + 2^2]^{1/2} \\
 &= \sqrt{30}
 \end{aligned}$$

$$b) \quad \vec{i}_{QP} = \frac{\vec{r}_{QP}}{r_{QP}} = \frac{\vec{i}_x - 5\vec{i}_y + 2\vec{i}_z}{\sqrt{30}}$$

$$c) \quad \vec{n} = n_x \vec{i}_x + n_y \vec{i}_y$$

$$\vec{n} \cdot \vec{i}_{QP} = 0 \rightarrow n_x - 5n_y = 0 \rightarrow \vec{n} = \frac{5\vec{i}_x + \vec{i}_y}{\sqrt{26}}$$

## Miscellaneous

$$26. \quad L \frac{di}{dt} + \frac{1}{C} \int i \, dt + iR = V_o$$

$$a) \quad \frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

$$b) \quad i = \hat{I} e^{st}$$

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

c) Initial conditions

$$i(t=0) = 0$$

$$L \left. \frac{di}{dt} \right|_{t=0} = V_o$$

$$\text{Steady state: } v_L = 0, v_c = V_o, v_R = 0$$

$$d) \quad \alpha = \frac{R}{2L}, \quad \beta = \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$s = -\alpha \pm j\beta$$

$$i(t) = [A_1 \sin \beta t + A_2 \cos \beta t] e^{-\alpha t}$$

$$\frac{di}{dt} = -\alpha [A_1 \sin \beta t + A_2 \cos \beta t] e^{-\alpha t} + \beta [A_1 \cos \beta t - A_2 \sin \beta t] e^{-\alpha t}$$

$$i(t=0) = 0 \rightarrow A_2 = 0$$

$$\left. \frac{di}{dt} \right|_{t=0} = \frac{V_o}{L} = -\alpha \frac{0}{2} + \beta A_1 \rightarrow A_1 = \frac{V_o}{L\beta}$$

$$i(t) = \frac{V_o}{L\beta} \sin\beta t e^{-\alpha t}$$

If  $\beta$  is imaginary

$$i(t) = \frac{V_o}{L} \frac{\sinh|\beta|t}{|\beta|} e^{-\alpha t}$$

If  $\beta = 0$

$$i(t) = \frac{V_o t}{L} e^{-\alpha t}$$

$$e) \quad V_R = iR = \frac{V_o R}{L\beta} \sin\beta t e^{-\alpha t}$$

$$\begin{aligned} V_C &= \frac{1}{C} \int i(t) dt = \frac{-V_o}{LC\beta} \frac{[\alpha \sin\beta t + \beta \cos\beta t] e^{-\alpha t}}{\alpha^2 + \beta^2} \\ &= \frac{-V_o}{\beta} [\alpha \sin\beta t + \beta \cos\beta t] e^{-\alpha t} + \text{constant} \end{aligned}$$

$$V_C(t=0) = 0 \rightarrow \text{constant} = V_o \rightarrow V_C = \frac{-V_o}{\beta} [\alpha \sin\beta t + \beta \cos\beta t] e^{-\alpha t} + V_o$$

$$V_L = L \frac{di}{dt} = \frac{V_o}{\beta} [\beta \cos\beta t - \alpha \sin\beta t] e^{-\alpha t}$$

$$f) \quad i(t) = [A_1 \sin\beta t + A_2 \cos\beta t] e^{-\alpha t}$$

$$\left. \begin{aligned} i(t=0) &= 0 \\ L \left. \frac{di}{dt} \right|_{t=0} &= -V_o \end{aligned} \right\} \rightarrow i(t) = \frac{-V_o}{L\beta} \sin\beta t e^{-\alpha t}$$

$$27. \quad a) \quad Lj\omega I_n + \frac{(I_n - I_{n+1})}{Cj\omega} - \frac{(I_{n-1} - I_n)}{Cj\omega} = 0$$

$$I_{n+1} - \left(2 - \frac{\omega^2}{\omega_o^2}\right) I_n + I_{n-1} = 0; \quad \omega_o^2 = \frac{1}{LC}$$

$$b) \quad I_n = \hat{I} \lambda^n$$

$$\hat{I} \left[ \lambda^{n+1} - \left(2 - \frac{\omega^2}{\omega_o^2}\right) \lambda^n + \lambda^{n-1} \right] = 0$$

$$\hat{I} \lambda^{n-1} \left[ \lambda^2 - \underbrace{\left( 2 - \frac{\omega^2}{2\omega_o^2} \right)}_0 \lambda + 1 \right] = 0$$

$$\lambda = \left( 1 - \frac{\omega^2}{2\omega_o^2} \right) \pm \sqrt{\left( 1 - \frac{\omega^2}{2\omega_o^2} \right)^2 - 1}$$

$$\lambda_1 = \left( 1 - \frac{\omega^2}{2\omega_o^2} \right) + \sqrt{\left( 1 - \frac{\omega^2}{2\omega_o^2} \right)^2 - 1}$$

$$\begin{aligned} \lambda_2 &= \left( 1 - \frac{\omega^2}{2\omega_o^2} \right) - \sqrt{\left( 1 - \frac{\omega^2}{2\omega_o^2} \right)^2 - 1} \\ &= \frac{1}{\lambda_1} \end{aligned}$$

$$c) \quad I_n = A \lambda_1^n + B \lambda_1^{-n}$$

$$V_n = \frac{I_n - I_{n+1}}{Cj\omega} = \frac{A \lambda_1^n (1 - \lambda_1) + B \lambda_1^{-n} (1 - \lambda_1^{-1})}{Cj\omega}$$

$$\text{At source end } (n=0) \rightarrow I_o = A + B$$

Open Circuited End

$$I_N = A \lambda_1^N + B \lambda_1^{-N} = 0 \rightarrow A = \frac{-I_o \lambda_1^{-2N}}{1 - \lambda_1^{-2N}}, \quad B = \frac{I_o}{1 - \lambda_1^{-2N}}$$

$$I_n = \frac{I_o}{1 - \lambda_1^{-2N}} [-\lambda_1^{n-2N} + \lambda_1^{-n}]$$

$$V_n = \frac{I_o}{Cj\omega(1 - \lambda_1^{-2N})} [-\lambda_1^{n-2N}(1 - \lambda_1) + \lambda_1^{-n}(1 - \lambda_1^{-1})]$$

Short Circuited End

$$V_N = \frac{A \lambda_1^N (1 - \lambda_1) + B \lambda_1^{-N} (1 - \lambda_1^{-1})}{Cj\omega} = 0$$

$$A = \frac{I_o \lambda_1^{-(N+1)}}{\lambda_1^{-(N+1)} + \lambda_1^N}, \quad B = \frac{I_o \lambda_1^N}{\lambda_1^{-(N+1)} + \lambda_1^N}$$

$$I_n = \frac{I_o}{\lambda_1^{-(N+1)} + \lambda_1^N} [\lambda_1^{n-N-1} + \lambda_1^{-n+N}]$$

$$V_n = \frac{I_o (1 - \lambda_1)}{Cj\omega(\lambda_1^{-(N+1)} + \lambda_1^N)} [\lambda_1^{n-N-1} - \lambda_1^{-n+N-1}]$$

d) If  $I_o = 0$ , for an open circuited end,  $I_n$  and  $V_n$  can be non-zero if

$$\lambda_1^{-2N} = 1 \rightarrow \lambda_1 = (1)^{\frac{1}{2N}} = e^{j2\pi r/2N} \quad r = 1, 2, \dots, 2N$$

$$\left(1 - \frac{\omega^2}{2\omega_o^2}\right) \pm \sqrt{\left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 1} = e^{j2\pi r/2N}$$

$$\pm \sqrt{\left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 1} = e^{j2\pi r/2N} - \left(1 - \frac{\omega^2}{2\omega_o^2}\right)$$

Square both sides and reduce

$$\left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 1 = e^{j4\pi r/2N} + \left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 2\left(1 - \frac{\omega^2}{2\omega_o^2}\right)e^{j2\pi r/2N}$$

$$\left(1 - \frac{\omega^2}{2\omega_o^2}\right) = \frac{e^{j2\pi r/2N} + e^{-j2\pi r/2N}}{2} = \cos \frac{\pi r}{N}$$

$$\omega^2 = 2\omega_o^2 \left(1 - \cos \frac{\pi r}{N}\right); \quad r = 1, 2, \dots, 2N$$

For a short circuited end

$$\lambda_1^N + \lambda_1^{-(N+1)} = 0 \rightarrow \lambda_1^{2N+1} = -1 \rightarrow \lambda_1 = (-1)^{1/(2N+1)} = e^{j\pi(2r-1)/(2N+1)}$$

$$r = 1, 2, \dots, 2N+1$$

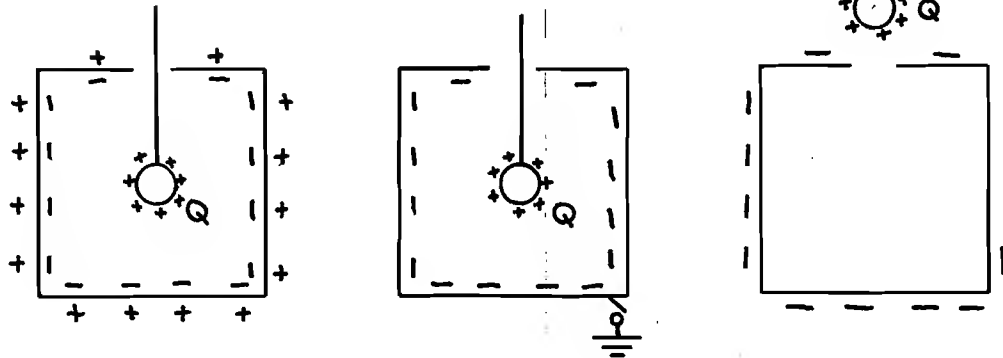
$$\rightarrow \omega^2 = 2\omega_o^2 \left(1 - \cos \frac{\pi(2r-1)}{2N+1}\right)$$

## CHAPTER 2 THE ELECTRIC FIELD

### Section 2.1

1.

a)



b) Net charge on pail after charged ball is removed =  $-Q$ .

2. a)  $\frac{Q}{2}$

b)  $q_1 = \frac{Q}{2}; q_2 = \frac{Q}{4}; q_3 = \frac{Q}{8}; \dots q_N = \frac{Q}{2^N}$   
Original sphere has charge  $q_N = \frac{Q}{2^N}$

$$\begin{aligned} \text{c) } Q_T &= Q \left[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^N} + \frac{1}{2^N} \right] \\ \frac{Q_T}{2} &= Q \left[ \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^N} + \frac{2}{2^{N+1}} \right] \end{aligned}$$

---


$$\frac{Q_T}{2} = Q \left[ \frac{1}{2} + \underbrace{\frac{1}{2^N} - \frac{2}{2^{N+1}}}_0 \right] \quad (\text{Subtract})$$

$$= \frac{Q}{2} \rightarrow Q_T = Q$$

The total charge in the system remains unchanged.

$$3. \quad q E_o = \rho_m g \frac{4}{3} \pi R^3 \rightarrow E_o = \frac{4}{3} \pi R^3 \frac{\rho_m g}{q}$$

$$4. \quad \frac{Q_1 Q_2}{4\pi\epsilon_o d^2} = T \sin\theta = \frac{Td}{2\ell} \text{ where } T \text{ is the tension in the strings.}$$

# THE ELECTRIC FIELD

$$T = \frac{Mg}{\cos\theta} = \frac{Mg\ell}{\sqrt{\ell^2 - \left(\frac{d}{2}\right)^2}}$$

$$Q_2 = \frac{2\pi\epsilon_0 d^3}{Q_1 \ell} T = \frac{2\pi\epsilon_0 d^3 Mg}{Q_1 \sqrt{\ell^2 - \left(\frac{d}{2}\right)^2}}$$

$$5. \quad a) \quad m\omega^2 R = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \rightarrow \omega = \left[ \frac{Q_1 Q_2}{4\pi\epsilon_0 R^3 m} \right]^{1/2}$$

$$b) \quad Q_1 = e, \quad Q_2 = Ze$$

$$L = m\omega R^2 = m\omega R^2 = nh/2\pi$$

$$m^2 \omega^2 R^4 = \left(\frac{nh}{2\pi}\right)^2 = \frac{m^2 R^4 Q_1 Q_2}{4\pi\epsilon_0 R^3 m} = \frac{m Q_1 Q_2 R}{4\pi\epsilon_0}$$

$$R = \frac{4\pi\epsilon_0 \left(\frac{nh}{2\pi}\right)^2}{m Q_1 Q_2} = \frac{4\pi\epsilon_0 \left(\frac{nh}{2\pi}\right)^2}{m Ze^2}$$

$$c) \quad \text{For } Z = 1, n = 1$$

$$R = \frac{\frac{4\pi}{36\pi} \times 10^{-9} \left(\frac{h}{2\pi}\right)^2}{9.1 \times 10^{-31} (1.6 \times 10^{-19})^2} = 5.3 \times 10^{-11} \text{ meters} \approx .53 \text{ \AA}$$

$$\omega = \left[ \frac{e^2}{4\pi\epsilon_0 R^3 m} \right]^{1/2} = \left[ \frac{(1.6 \times 10^{-19})^2}{\frac{10^{-9}}{9} (5.3 \times 10^{-11})^3 (9.1 \times 10^{-31})} \right]^{1/2} = 4.1 \times 10^{16} \text{ rad/sec}$$

$$v = \omega R = 2.2 \times 10^6 \text{ m/sec}$$

$$6. \quad T = \frac{Mg}{\cos\theta} = \frac{Q_1 Q_2}{4\pi\epsilon_0 s^2 \sin\theta} \quad \text{where} \quad \sin\theta = \frac{s}{2\ell} \quad (Q_1 = Q_2 = \frac{Q}{2})$$

$$\frac{Mg 4\pi\epsilon_0 (2\ell \sin\theta)^2 \sin\theta}{Q_1 Q_2 \cos\theta} = 1$$

$$\sin^2\theta \tan\theta = \frac{Q_1 Q_2}{16\pi\epsilon_0 \ell^2 Mg} = \frac{Q^2}{64\pi\epsilon_0 \ell^2 Mg}$$



# THE ELECTRIC FIELD

$$7. \quad a) \quad m_1 \frac{d^2 r_1}{dt^2} = - \frac{q_1 q_2}{4\pi\epsilon_0 (r_2 - r_1)^2} \quad (\text{multiply by } m_2)$$

$$m_2 \frac{d^2 r_2}{dt^2} = \frac{q_1 q_2}{4\pi\epsilon_0 (r_2 - r_1)^2} \quad (\text{multiply by } m_1)$$

$$\frac{m_1 m_2}{m_1 m_2} \frac{d^2 (r_2 - r_1)}{dt^2} = \frac{q_1 q_2 (m_2 + m_1)}{4\pi\epsilon_0 (r_2 - r_1)^2} \quad (\text{subtract})$$

$$m \frac{d^2 r}{dt^2} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}; \quad m = \frac{m_1 m_2}{m_1 + m_2}, \quad r = r_2 - r_1$$

$$b) \quad m \frac{dv}{dt} = m \frac{dv}{dr} \frac{dr}{dt} = mv \frac{dv}{dr} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$$

$$v dv = \frac{q_1 q_2}{4\pi\epsilon_0 m r^2} dr \rightarrow \frac{1}{2} v^2 = - \frac{q_1 q_2}{4\pi\epsilon_0 m} \left( \frac{1}{r} - \frac{1}{r_0} \right)$$

$$v = \pm \sqrt{\frac{-q_1 q_2}{2\pi\epsilon_0 m} \left( \frac{1}{r} - \frac{1}{r_0} \right)} \quad \begin{array}{l} q_1 q_2 > 0, \quad r > r_0, \quad v > 0 \\ q_1 q_2 < 0, \quad r < r_0, \quad v < 0 \end{array}$$

$$c) \quad v = \frac{dr}{dt} = \pm \sqrt{\frac{-q_1 q_2}{2\pi\epsilon_0 m} \left( \frac{1}{r} - \frac{1}{r_0} \right)}$$

$$\rightarrow \frac{dr}{\left[ \frac{1}{r} - \frac{1}{r_0} \right]^{1/2}} = \frac{\sqrt{r} dr}{\left[ 1 - \frac{r}{r_0} \right]^{1/2}} = \pm \left[ \frac{-q_1 q_2}{2\pi m \epsilon_0} \right]^{1/2} dt$$

$$i) \quad q_1 q_2 < 0 \quad (\text{charges attract})$$

$$\sqrt{r_0} \left[ -\sqrt{r} \sqrt{r_0 - r} + r_0 \sin^{-1} \sqrt{\frac{r}{r_0}} \right] = - \left[ \frac{-q_1 q_2}{2\pi m \epsilon_0} \right]^{1/2} t + \text{constant}$$

$$\sqrt{r_0} \left[ -\sqrt{r} \sqrt{r_0 - r} + r_0 \sin^{-1} \sqrt{\frac{r}{r_0}} - \frac{\pi}{2} \right] = - \left[ \frac{-q_1 q_2}{2\pi m \epsilon_0} \right]^{1/2} t$$

$$r(t=0) = r_0 \rightarrow \text{constant} = \frac{\pi r_0^{3/2}}{2}$$

# THE ELECTRIC FIELD

ii)  $q_1 q_2 > 0$  (charges repel)

$$\sqrt{r_o} \left[ \sqrt{r} \sqrt{r-r_o} + r_o \ln \left[ \sqrt{r} + \sqrt{r-r_o} \right] \right] = \left[ \frac{q_1 q_2}{2\pi m \epsilon_o} \right]^{1/2} t + \text{constant}$$

$$\sqrt{r_o} \left[ \sqrt{r} \sqrt{r-r_o} + r_o \ln \left[ \sqrt{\frac{r}{r_o}} + \sqrt{\frac{r}{r_o} - 1} \right] \right] = \left[ \frac{q_1 q_2}{2\pi m \epsilon_o} \right]^{1/2} t$$

$$r(t=0) = r_o \rightarrow \text{constant} = r_o^{3/2} \ln \sqrt{r_o}$$

d) If  $q_1 q_2 < 0$  see (c)(i)

$r = 0$  when

$$\left[ \frac{-q_1 q_2}{2\pi m \epsilon_o} \right]^{1/2} t = \frac{\pi}{2} r_o^{3/2} \rightarrow t = \frac{\pi}{2} r_o^{3/2} \left[ \frac{2\pi m \epsilon_o}{-q_1 q_2} \right]^{1/2}$$

$$e) \beta_1 \frac{dr_1}{dt} = - \frac{q_1 q_2}{4\pi \epsilon_o (r_2 - r_1)^2} \quad (\text{multiply by } \beta_2)$$

$$\beta_2 \frac{dr_2}{dt} = \frac{q_1 q_2}{4\pi \epsilon_o (r_2 - r_1)^2} \quad (\text{multiply by } \beta_1)$$

$$\beta_2 \beta_1 \frac{d}{dt} (r_2 - r_1) = \frac{q_1 q_2}{4\pi \epsilon_o (r_2 - r_1)^2} (\beta_2 + \beta_1) \quad (\text{subtract})$$

$$\beta \frac{dr}{dt} = \frac{q_1 q_2}{4\pi \epsilon_o r^2}; \quad \beta = \frac{\beta_1 \beta_2}{\beta_1 + \beta_2}, \quad r = r_2 - r_1$$

$$r^2 dr = \frac{q_1 q_2}{4\pi \epsilon_o \beta} dt$$

$$\frac{r^3}{3} = \frac{q_1 q_2}{4\pi \epsilon_o \beta} t + \frac{r_o^3}{3}$$

If  $q_1 q_2 < 0$ ,  $r = 0$  when

$$\frac{-q_1 q_2 t}{4\pi \epsilon_o \beta} = \frac{r_o^3}{3} \rightarrow t = - \frac{4\pi \epsilon_o \beta r_o^3}{3 q_1 q_2}$$

$$8. \quad m \frac{d^2 z}{dt^2} = qE_o \rightarrow z = \frac{qE_o t^2}{2m} + \cancel{v_{zo} t} + \cancel{z_o}$$

$$m \frac{d^2 x}{dt^2} = 0 \rightarrow x = v_o t \rightarrow t = \frac{x}{v_o}$$

$$z = \frac{qE_o}{2m} \frac{x^2}{v_o^2}$$

$$z(x=L) = h = \frac{qE_o L^2}{2mv_o^2}$$

$$9. \quad m \frac{d^2 \xi}{dt^2} = -mg \underbrace{\sin \theta}_{\approx \frac{\xi}{\ell}} + \frac{qQ}{4\pi\epsilon_o} \left[ \frac{1}{(D + \xi)^2} - \frac{1}{(D - \xi)^2} \right]$$

$$\approx \frac{-mg\xi}{\ell} + \frac{qQ}{4\pi\epsilon_o D^2} \left\{ \frac{1}{(1 + \frac{\xi}{D})^2} - \frac{1}{(1 - \frac{\xi}{D})^2} \right\}$$

$$\approx \frac{-mg\xi}{\ell} + \frac{qQ}{4\pi\epsilon_o D^2} \left\{ 1 - \frac{2\xi}{D} - (1 + \frac{2\xi}{D}) \right\}$$

$$= \frac{-mg\xi}{\ell} - \frac{qQ}{\pi\epsilon_o D^3} \xi$$

$$m \frac{d^2 \xi}{dt^2} = -\left(\frac{mg}{\ell} + \frac{qQ}{\pi\epsilon_o D^3}\right) \xi$$

$$\xi = A_1 \sin \omega_o t + A_2 \cos \omega_o t; \quad \omega_o^2 = \frac{g}{\ell} + \frac{qQ}{m\pi\epsilon_o D^3}$$

Unstable when

$$\omega_o^2 < 0 \rightarrow qQ < \frac{-mg}{\ell} \pi\epsilon_o D^3$$

$$\text{If } \xi(t=0) = \xi_o, \quad \frac{\partial \xi}{\partial t}(t=0) = 0$$

$$\xi = \xi_o \cos \omega_o t$$

# THE ELECTRIC FIELD

10. a)  $\vec{F}_q = \frac{qQ}{4\pi\epsilon_0 a^2} \vec{i}_y$

b)  $\vec{E}_p = \frac{1}{4\pi\epsilon_0} \left[ \frac{2Q}{\left(\frac{7}{16}\right)^{3/2} a^2} \frac{\sqrt{3}}{4} - \frac{q}{\frac{3}{16} a^2} \right] \vec{i}_y$   
 $= \frac{4}{\pi\epsilon_0 a^2} \left[ \frac{2\sqrt{3} Q}{(7)^{3/2}} - \frac{q}{3} \right] \vec{i}_y$

$\vec{E}_p = 0 \rightarrow q = \frac{6\sqrt{3}}{7^{3/2}} Q$

11.  $\vec{E}_1 = \frac{q_1 \left[ -\frac{a}{2} \vec{i}_x + \frac{a}{2} \vec{i}_y + z \vec{i}_z \right]}{4\pi\epsilon_0 \left[ z^2 + \frac{a^2}{2} \right]^{3/2}}$

$\vec{E}_2 = \frac{q_2 \left[ \frac{a}{2} \vec{i}_x + \frac{a}{2} \vec{i}_y + z \vec{i}_z \right]}{4\pi\epsilon_0 \left[ z^2 + \frac{a^2}{2} \right]^{3/2}}$

$\vec{E}_3 = \frac{q_3 \left[ \frac{a}{2} \vec{i}_x - \frac{a}{2} \vec{i}_y + z \vec{i}_z \right]}{4\pi\epsilon_0 \left[ z^2 + \frac{a^2}{2} \right]^{3/2}}$

$\vec{E}_4 = \frac{q_4 \left[ -\frac{a}{2} \vec{i}_x - \frac{a}{2} \vec{i}_y + z \vec{i}_z \right]}{4\pi\epsilon_0 \left[ z^2 + \frac{a^2}{2} \right]^{3/2}}$

$\vec{E}_T(z) = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4$

a)  $q_1 = q_2 = q_3 = q_4 \equiv q \rightarrow \vec{E}_T(z) = \frac{4qz}{4\pi\epsilon_0 \left( z^2 + \frac{a^2}{2} \right)^{3/2}} \vec{i}_z$

b)  $q_1 = q_3 \equiv q, q_2 = q_4 \equiv -q \rightarrow \vec{E}_T(z) = 0$

c)  $q_1 = q_2 \equiv q, q_3 = q_4 \equiv -q \rightarrow \vec{E}_T(z) = \frac{2qa}{4\pi\epsilon_0 \left( z^2 + \frac{a^2}{2} \right)^{3/2}} \vec{i}_y$

$$12. \quad a) \quad q = \int_{-\infty}^0 \lambda_o e^{z/a} dz + \int_0^{\infty} \lambda_o e^{-z/a} dz$$

$$= \lambda_o a e^{z/a} \Big|_{-\infty}^0 - \lambda_o a e^{-z/a} \Big|_0^{\infty}$$

$$= 2\lambda_o a$$

$$b) \quad q = 4\pi \int_0^{\infty} \frac{\rho_o r^2 dr}{\left[1 + \frac{r}{a}\right]^4}$$

$$\text{Let } u = 1 + \frac{r}{a}; \quad du = \frac{1}{a} dr$$

$$q = 4\pi \int_1^{\infty} \frac{\rho_o a^3 (u-1)^2}{u^4} du$$

$$= 4\pi \rho_o a^3 \int_1^{\infty} \frac{u^2 - 2u + 1}{u^4} du$$

$$= 4\pi \rho_o a^3 \int_1^{\infty} \left[ \frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4} \right] du$$

$$= 4\pi \rho_o a^3 \left[ -\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} \right] \Big|_1^{\infty}$$

$$= \frac{4}{3} \pi \rho_o a^3$$

$$c) \quad q = \int_{y=-\infty}^{+\infty} \int_{x=0}^{\infty} \frac{2\sigma_o e^{-x/a}}{\left[1 + \left(\frac{y}{b}\right)^2\right]} dx dy$$

$$= \int_{y=-\infty}^{+\infty} \frac{-2\sigma_o a e^{-x/a} \Big|_0^{\infty}}{\left[1 + \left(\frac{y}{b}\right)^2\right]} dy$$

$$= 2\sigma_o a \int_{-\infty}^{+\infty} \frac{dy}{\left[1 + \left(\frac{y}{b}\right)^2\right]}$$

$$= 2\sigma_o a b \tan^{-1} \frac{y}{b} \Big|_{-\infty}^{+\infty}$$

$$= 2\sigma_o a b \pi$$

# THE ELECTRIC FIELD

13. a)  $M \frac{d^2 x}{dt^2} = \frac{q\sigma_o}{2\epsilon_o} - Mg$

$$x = \left( \frac{q\sigma_o}{2\epsilon_o M} - g \right) \frac{t^2}{2} + v_o t + x_o$$

b) Charge remains stationary if

$$\frac{q\sigma_o}{2M\epsilon_o} - g = 0 \rightarrow \sigma_o = \frac{2Mg\epsilon_o}{q}$$

c)  $x = 0$  when  $\left( \frac{q\sigma_o}{2M\epsilon_o} - g \right) \frac{t^2}{2} = -x_o$

$$t = \left[ \frac{2x_o}{g - \frac{q\sigma_o}{2M\epsilon_o}} \right]^{1/2}$$

$$v = \frac{dx}{dt} = \left( \frac{q\sigma_o}{2M\epsilon_o} - g \right) t = - \left[ 2x_o \left[ g - \frac{q\sigma_o}{2M\epsilon_o} \right] \right]^{1/2}$$

14. a)  $\vec{F}_q = \frac{q\lambda_o}{2\pi\epsilon_o D} \vec{i}_y$

$$\begin{aligned} \text{b) } \vec{F}_\ell &= - \int_{-\infty}^{+\infty} \frac{\lambda_o dz q D}{4\pi\epsilon_o (D^2 + z^2)^{3/2}} \vec{i}_y \\ &= \frac{-q\lambda_o D}{4\pi\epsilon_o} \int_{-\infty}^{+\infty} \frac{dz}{(D^2 + z^2)^{3/2}} \vec{i}_y \\ &= \frac{-q\lambda_o D}{4\pi\epsilon_o} \frac{z}{D^2 \sqrt{D^2 + z^2}} \Big|_{-\infty}^{+\infty} \vec{i}_y \\ &= \frac{-q\lambda_o}{2\pi\epsilon_o D} \vec{i}_y \end{aligned}$$

$$\begin{aligned} \text{c) } \vec{F}_q &= -\vec{F}_\ell = \frac{q}{4\pi\epsilon_o} \int_0^\infty \frac{2\lambda_o z D dz}{a(D^2 + z^2)^{3/2}} \vec{i}_y \\ &= \frac{\lambda_o D q}{2\pi\epsilon_o a} \int_0^\infty \frac{z dz}{(D^2 + z^2)^{3/2}} \vec{i}_y \\ &= \frac{\lambda_o D q}{2\pi\epsilon_o a} \frac{-1}{\sqrt{D^2 + z^2}} \Big|_0^\infty \vec{i}_y \end{aligned}$$

# THE ELECTRIC FIELD

$$= \frac{\lambda_o q}{2\pi\epsilon_o a} \bar{i}_y$$

$$15. \quad T = \frac{Mg}{\cos\theta} = \frac{Q\sigma_o}{2\epsilon_o \sin\theta} \rightarrow \tan\theta = \frac{Q\sigma_o}{2\epsilon_o Mg}$$

$$16. \quad a) \quad dE_r = \frac{\lambda \, dz \, r}{4\pi\epsilon_o (r^2 + z^2)^{3/2}}$$

$$E_r = \frac{\lambda r}{4\pi\epsilon_o} \int_{-L}^L \frac{dz}{(r^2 + z^2)^{3/2}}$$

$$= \frac{\lambda r}{4\pi\epsilon_o} \left. \frac{z}{r^2 \sqrt{z^2 + r^2}} \right|_{-L}^{+L}$$

$$= \frac{\lambda}{2\pi\epsilon_o r} \frac{L}{\sqrt{L^2 + r^2}}$$

$$b) \quad dE_x = \frac{\sigma_o \, dy}{2\pi\epsilon_o \sqrt{x^2 + y^2}} \frac{Lx}{\sqrt{L^2 + x^2 + y^2} \sqrt{x^2 + y^2}} \quad ; y > 0$$

$$= \frac{\sigma_o Lx \, dy}{2\pi\epsilon_o (x^2 + y^2) \sqrt{L^2 + x^2 + y^2}}$$

$$E_x = \frac{\sigma_o Lx}{2\pi\epsilon_o} \int_{-\infty}^{+\infty} \frac{dy}{(x^2 + y^2) \sqrt{L^2 + x^2 + y^2}}$$

$$\text{Let } u = x^2 + y^2, \quad y = \sqrt{u - x^2}; \quad du = 2y \, dy = 2\sqrt{u - x^2} \, dy$$

$$E_x = \frac{\sigma_o Lx}{2\pi\epsilon_o} \int_{x^2}^{\infty} \frac{du}{u \sqrt{u - x^2} \sqrt{L^2 + u}}$$

$$= \frac{\sigma_o Lx}{2\pi\epsilon_o} \int_{x^2}^{\infty} \frac{du}{u \sqrt{u^2 + u(L^2 - x^2) - L^2 x^2}}$$

$$= \frac{\sigma_o Lx}{2\pi\epsilon_o} \frac{1}{Lx} \sin^{-1} \frac{(L^2 - x^2)u - 2L^2 x^2}{u \sqrt{(L^2 - x^2)^2 + 4L^2 x^2}} \Big|_{u=x^2}^{\infty}$$

$$= \frac{\sigma_o}{2\pi\epsilon_o} \sin^{-1} \frac{(L^2 - x^2)u - 2L^2 x^2}{u(L^2 + x^2)} \Big|_{u=x^2}^{\infty}$$

# THE ELECTRIC FIELD

$$= \frac{\sigma_o}{2\pi\epsilon_o} \left[ \sin^{-1} \frac{(L^2 - x^2)}{L^2 + x^2} + \frac{\pi}{2} \right]$$

$$\lim_{L \rightarrow \infty} E_x = \frac{\sigma_o}{2\epsilon_o}$$

17. a)  $dE_y = \frac{-\sigma_o R d\phi}{2\pi\epsilon_o R} \sin\phi$

$$E_y = \frac{-\sigma_o}{2\pi\epsilon_o} \int_0^\pi \sin\phi d\phi$$

$$= \frac{+\sigma_o}{2\pi\epsilon_o} \cos\phi \Big|_0^\pi$$

$$= \frac{-\sigma_o}{\pi\epsilon_o}$$

b)  $dE_y = \frac{-\rho_o dr}{\pi\epsilon_o}$

$$E_y = \frac{-\rho_o}{\pi\epsilon_o} \int_{r=0}^R dr$$

$$= \frac{-\rho_o}{\pi\epsilon_o} r \Big|_{r=0}^R$$

$$= \frac{-\rho_o R}{\pi\epsilon_o}$$

c) Surface charged hemi-sphere

$$dE_z = \frac{-\sigma_o}{4\pi\epsilon_o} \sin\theta \cos\theta d\theta d\phi$$

$$E_z = \frac{-\sigma_o}{4\pi\epsilon_o} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta d\theta d\phi$$

$$= \frac{-\sigma_o}{2\epsilon_o} \int_{\theta=0}^{\pi/2} \frac{\sin 2\theta}{2} d\theta$$



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$$= \frac{+\sigma}{8\epsilon_0} \cos 2\theta \Big|_0^{\pi/2}$$

$$= \frac{-\sigma}{4\epsilon_0}$$

Volume charged hemi-sphere of radius R

$$dE_z = \frac{-\rho_0 dr}{4\epsilon_0}$$

$$E_z = \frac{-\rho_0}{4\epsilon_0} \int_0^R dr = \frac{-\rho_0 R}{4\epsilon_0}$$

$$\begin{aligned} 18. \quad a) \quad \vec{E} &= \frac{\lambda_0 a}{4\pi\epsilon_0 [z^2 + a^2]} \left\{ \int_0^\pi \frac{(z\vec{i}_z - a\vec{i}_r) d\phi}{\sqrt{z^2 + a^2}} - \int_\pi^{2\pi} \frac{(z\vec{i}_z - a\vec{i}_r) d\phi}{\sqrt{z^2 + a^2}} \right\} \\ &= \frac{-\lambda_0 a^2}{4\pi\epsilon_0 [z^2 + a^2]^{3/2}} \left\{ \int_0^\pi [\cos\phi \vec{i}_x + \sin\phi \vec{i}_y] d\phi - \int_\pi^{2\pi} [\cos\phi \vec{i}_x + \sin\phi \vec{i}_y] d\phi \right\} \\ \vec{E} &= \frac{-\lambda_0 a^2}{\pi\epsilon_0 [z^2 + a^2]^{3/2}} \vec{i}_y \end{aligned}$$

$$\begin{aligned} b) \quad dE_y &= \frac{-\sigma_0 r^2 dr}{\pi\epsilon_0 [z^2 + r^2]^{3/2}} ; \quad E_y = \frac{-\sigma_0}{\pi\epsilon_0} \int_0^a \frac{r^2 dr}{[z^2 + r^2]^{3/2}} \\ &= \frac{-\sigma_0}{\pi\epsilon_0} \left[ \frac{-r}{\sqrt{z^2 + r^2}} + \ln(r + \sqrt{z^2 + r^2}) \right]_0^a \\ &= \frac{-\sigma_0}{\pi\epsilon_0} \left[ \frac{-a}{\sqrt{z^2 + a^2}} + \ln \left( \frac{a + \sqrt{z^2 + a^2}}{|z|} \right) \right] \end{aligned}$$

# THE ELECTRIC FIELD

$$19. a) \quad dE_z = \frac{\lambda_o dxz}{\pi\epsilon_o (x^2 + \frac{a^2}{4} + z^2)^{3/2}}$$

$$E_z = \frac{\lambda_o z}{\pi\epsilon_o} \int_{-a/2}^{+a/2} \frac{dx}{(x^2 + \frac{a^2}{4} + z^2)^{3/2}}$$

$$= \frac{\lambda_o z}{\pi\epsilon_o} \frac{x}{(\frac{a^2}{4} + z^2) \sqrt{x^2 + \frac{a^2}{4} + z^2}} \Big|_{x=-a/2}^{+a/2}$$

$$= \frac{\lambda_o za}{\pi\epsilon_o (\frac{a^2}{4} + z^2) \sqrt{\frac{a^2}{4} + z^2}}$$

Check

$$\lim_{z \gg a} E_z = \pm \frac{\lambda_o a}{\pi\epsilon_o z^2} = \pm \frac{Q_T}{4\pi\epsilon_o z^2} \quad \begin{matrix} z > 0 \\ z < 0 \end{matrix} \quad (Q_T = 4\lambda_o a) \quad \leftarrow \text{field of point charge}$$

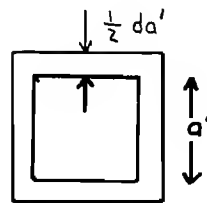
$$b) \quad E_z = \int_0^a \frac{\sigma_o z a' \frac{da'}{2}}{\pi\epsilon_o (\frac{a'^2}{4} + z^2) \sqrt{\frac{a'^2}{4} + z^2}}$$

$$u = z^2 + \frac{a'^2}{4}; \quad du = \frac{a' da'}{2}$$

$$dE_z = \frac{\sigma_o z du}{\pi\epsilon_o u \sqrt{2u - z^2}}$$

$$E_z = \frac{\sigma_o z}{\pi\epsilon_o} \int_{z^2}^{z^2 + a^2/4} \frac{du}{u \sqrt{2u - z^2}}$$

$$= \frac{\sigma_o z}{\pi\epsilon_o} \frac{2}{|z|} \tan^{-1} \sqrt{\frac{2u - z^2}{z^2}} \Big|_{u=z^2}^{z^2 + a^2/4}$$



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$$= + \frac{2\sigma_o}{\pi\epsilon_o} \left[ \tan^{-1} \sqrt{\frac{z^2 + a^2/2}{z^2}} - \underbrace{\tan^{-1} 1}_{\pi/4} \right] \begin{matrix} z > 0 \\ z < 0 \end{matrix}$$

$$\lim_{a \rightarrow \infty} E_z = \pm \frac{2\sigma_o}{\pi\epsilon_o} \frac{\pi}{4} = \pm \frac{\sigma_o}{2\epsilon_o} \quad \begin{matrix} z > 0 \\ z < 0 \end{matrix} \quad \leftarrow \text{field of sheet of surface charge}$$

$$20. \quad a) \quad dE_y = \frac{-2\lambda_o a^2 \sin\phi \, d\phi}{4\pi\epsilon_o (a^2 + z^2)^{3/2}}$$

$$E_y = \frac{-2\lambda_o a^2}{4\pi\epsilon_o (a^2 + z^2)^{3/2}} \int_0^\pi \sin\phi \, d\phi = \frac{-\lambda_o a^2}{\pi\epsilon_o (a^2 + z^2)^{3/2}}$$

$$b) \quad dE_y = \frac{-\sigma_o r^2 \, dr}{\pi\epsilon_o (r^2 + z^2)^{3/2}}$$

$$E_y = \frac{-\sigma_o}{\pi\epsilon_o} \int_0^a \frac{r^2 \, dr}{(r^2 + z^2)^{3/2}} = \frac{-\sigma_o}{\pi\epsilon_o} \left[ \frac{-r}{\sqrt{r^2 + z^2}} + \ln[r + \sqrt{r^2 + z^2}] \right] \Big|_{r=0}^a$$

$$= \frac{\sigma_o}{\pi\epsilon_o} \left\{ \frac{a}{\sqrt{a^2 + z^2}} - \ln \left[ \frac{a + \sqrt{a^2 + z^2}}{|z|} \right] \right\}$$

$$c) \quad E_y = \frac{-\lambda_o a^2}{4\pi\epsilon_o (a^2 + z^2)^{3/2}} \underbrace{\int_0^{2\pi} \sin^2\phi \, d\phi}_{\pi}$$

$$= \frac{-\lambda_o a^2}{4\epsilon_o (a^2 + z^2)^{3/2}}$$

$$d) \quad E_y = \frac{\sigma_o}{4\epsilon_o} \left\{ \frac{a}{\sqrt{a^2 + z^2}} - \ln \left[ \frac{a + \sqrt{a^2 + z^2}}{|z|} \right] \right\}$$

$$21. \quad \overline{dE} = \frac{2\lambda_o}{4\pi\epsilon_o} \frac{(z\overline{i}_z - y\overline{i}_y)}{[a^2 + y^2 + z^2]^{3/2}} \, dy + \frac{\lambda_o a \, d\phi [-a\overline{i}_r + z\overline{i}_z]}{4\pi\epsilon_o (z^2 + a^2)^{3/2}}$$

$$E_z = \frac{\lambda_o z}{2\pi\epsilon_o} \int_{-\infty}^0 \frac{dy}{[a^2 + y^2 + z^2]^{3/2}} + \frac{\lambda_o a z}{4\pi\epsilon_o (z^2 + a^2)^{3/2}} \int_0^\pi d\phi$$

# THE ELECTRIC FIELD

$$\begin{aligned}
 &= \frac{\lambda_o z}{2\pi\epsilon_o} \frac{y}{(a^2 + z^2)(a^2 + y^2 + z^2)^{1/2}} \Big|_{y=-\infty}^0 + \frac{\lambda_o az}{4\epsilon_o (z^2 + a^2)^{3/2}} \\
 &= \frac{\lambda_o z}{2\pi\epsilon_o (a^2 + z^2)} \left[ 1 + \frac{\pi a}{2(a^2 + z^2)^{1/2}} \right] \\
 E_y &= \frac{-\lambda_o}{2\pi\epsilon_o} \int_{-\infty}^0 \frac{y dy}{[a^2 + y^2 + z^2]^{3/2}} - \frac{-\lambda_o a^2}{4\pi\epsilon_o (z^2 + a^2)^{3/2}} \int_0^\pi \sin\phi d\phi \\
 &= \frac{+\lambda_o}{2\pi\epsilon_o} \frac{1}{[a^2 + y^2 + z^2]^{1/2}} \Big|_{y=-\infty}^0 + \frac{\lambda_o a^2}{4\pi\epsilon_o (z^2 + a^2)^{3/2}} \cos\phi \Big|_0^\pi \\
 &= \frac{\lambda_o}{2\pi\epsilon_o (a^2 + z^2)^{1/2}} \left[ 1 - \frac{a^2}{(z^2 + a^2)} \right] = \frac{\lambda_o z^2}{2\pi\epsilon_o (a^2 + z^2)^{3/2}}
 \end{aligned}$$

## Section 2.4

22. a)  $\vec{E} = ar^2 \vec{i}_r \rightarrow \nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = 4Ar = \frac{\rho}{\epsilon_o}$

$$\begin{aligned}
 Q_T &= \int_V \rho dV \\
 &= \int_V 4\epsilon_o Ar^3 \sin\theta dr d\theta d\phi \\
 &= 16\pi\epsilon_o A \int_0^R r^3 dr \\
 &= 4\pi\epsilon_o AR^4
 \end{aligned}$$

## Another Method

$$Q_T = \int_S \epsilon_o \vec{E} \cdot d\vec{S} = \epsilon_o E_r (r=R) 4\pi R^2 = 4\pi\epsilon_o AR^4$$

b)  $\vec{E} = Ar^2 \vec{i}_r \rightarrow \nabla \cdot \vec{E} = \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = 3Ar = \frac{\rho}{\epsilon_o}$

$$\begin{aligned}
 Q_T &= \int_V \rho dV \\
 &= \int_V 3\epsilon_o Ar^2 dr d\phi dz
 \end{aligned}$$

# THE ELECTRIC FIELD

$$= 6\pi\epsilon_0 AL \int_0^a r^2 dr$$

$$= 2\pi\epsilon_0 ALa^3$$

$$c) \quad \vec{E} = A(x\vec{i}_x + y\vec{i}_y) \rightarrow \nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 2A = \frac{\rho}{\epsilon_0}$$

$$Q_T = \rho a^3 = 2\epsilon_0 Aa^3$$

$$23. a) \quad \epsilon_0 \frac{dE_x}{dx} = \rho(x) = \begin{cases} \rho_0 e^{-x/a} & x > 0 \\ \rho_0 e^{+x/a} & x < 0 \end{cases}$$

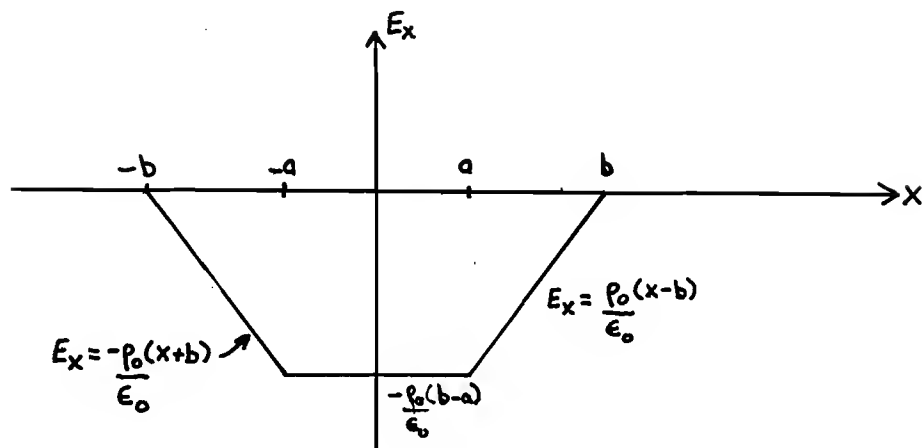
Integrating

$$E_x = \begin{cases} \frac{-\rho_0 a}{\epsilon_0} e^{-x/a} + C_1 & x > 0 \\ \frac{\rho_0 a}{\epsilon_0} e^{+x/a} + C_2 & x < 0 \end{cases}$$

$$\text{By symmetry, } E_x(x=0) = 0 \rightarrow C_1 = \frac{\rho_0 a}{\epsilon_0}, C_2 = \frac{-\rho_0 a}{\epsilon_0}$$

$$E_x = \begin{cases} \frac{-\rho_0 a}{\epsilon_0} (e^{-x/a} - 1) & x > 0 \\ \frac{\rho_0 a}{\epsilon_0} (e^{x/a} - 1) & x < 0 \end{cases}$$

b)



# THE ELECTRIC FIELD

$$c) \quad \epsilon_0 \frac{dE_x}{dx} = \rho(x) = \frac{\rho_0 x}{d} \rightarrow E_x = \begin{cases} \frac{\rho_0}{2\epsilon_0 d} (x^2 - d^2) & |x| < d \\ 0 & |x| > d \end{cases}$$

$$d) \quad \rho(x) = \epsilon_0 \frac{dE_x}{dx} = \begin{cases} \rho_0 (1 + \frac{x}{d}) & -d < x < 0 \\ \rho_0 (1 - \frac{x}{d}) & 0 < x < d \\ 0 & |x| > d \end{cases}$$

Integrating

$$E_x = \begin{cases} \frac{\rho_0 x}{\epsilon_0} (1 + \frac{x}{2d}) & -d < x < 0 \\ \frac{\rho_0 x}{\epsilon_0} (1 - \frac{x}{2d}) & 0 < x < d \\ + \frac{\rho_0 d}{2\epsilon_0} & x > d \\ - \frac{\rho_0 d}{2\epsilon_0} & x < -d \end{cases}$$

$$24. a) \quad \frac{\epsilon_0}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = \rho(r) = \rho_0 e^{-r/a}$$

$$\frac{\partial}{\partial r} (r^2 E_r) = \frac{\rho_0 r^2}{\epsilon_0} e^{-r/a} \rightarrow r^2 E_r = \frac{-\rho_0 a}{\epsilon_0} [r^2 + 2a^2 (\frac{r}{a} + 1)] e^{-r/a} + \text{constant}$$

$$\text{No point charge at } r = 0 \rightarrow \text{constant} = \frac{2a^3 \rho_0}{\epsilon_0}$$

$$E_r = \frac{-\rho_0 a}{\epsilon_0} \left\{ \left[ 1 + \frac{2a}{r} + \frac{2a^2}{r^2} \right] e^{-r/a} - \frac{2a^2}{r^2} \right\}$$

b) Use Gaussian sphere of radius r

$$\oint_S \epsilon_0 \vec{E} \cdot d\vec{S} = 4\pi \epsilon_0 r^2 E_r = Q_{\text{enclosed}} = \begin{cases} \frac{4}{3} \pi r^3 \rho_1 & r < R_1 \\ \frac{4}{3} \pi R_1^3 \rho_1 + \frac{4}{3} \pi (r^3 - R_1^3) \rho_2 & R_1 < r < R_2 \\ \frac{4}{3} \pi [\rho_1 R_1^3 + (R_2^3 - R_1^3) \rho_2] & r > R_2 \end{cases}$$

# THE ELECTRIC FIELD

$$E_r = \begin{cases} \frac{\rho_1 r}{3\epsilon_0} & r < R_1 \\ \frac{\rho_1 R_1^3 + (r^3 - R_1^3)\rho_2}{3\epsilon_0 r^2} & R_1 < r < R_2 \\ \frac{[\rho_1 R_1^3 + (R_2^3 - R_1^3)\rho_2]}{3\epsilon_0 r^2} & r > R_2 \end{cases}$$

c) Use Gaussian sphere of radius  $r$

$$\oint_S \epsilon_0 \vec{E} \cdot d\vec{S} = 4\pi\epsilon_0 r^2 E_r = Q_{\text{enclosed}} = \begin{cases} \frac{4\pi\rho_0}{R} \int_0^r r'^3 dr' = \frac{\pi\rho_0 r^4}{R} & r < R \\ \frac{4\pi\rho_0}{R} \int_0^R r'^3 dr' = \pi\rho_0 R^3 & r > R \end{cases}$$

$$E_r = \begin{cases} \frac{\rho_0 r^2}{4\epsilon_0 R} & r < R \\ \frac{\rho_0 R^3}{4\epsilon_0 r^2} & r > R \end{cases}$$

25. a)  $\frac{1}{r} \frac{\partial}{\partial r} (rE_r) = \frac{\rho_0}{\epsilon_0} e^{-r/a}$

$$rE_r = \frac{\rho_0}{\epsilon_0} \int r e^{-r/a} dr + \text{constant}$$

$$= \frac{\rho_0 a^2}{\epsilon_0} e^{-r/a} \left(-\frac{r}{a} - 1\right) + \text{constant}$$

$$E_r = \frac{-\rho_0 a^2}{\epsilon_0} \left[ e^{-r/a} \left(\frac{1}{a} + \frac{1}{r}\right) - \frac{1}{r} \right] \quad (E_r(r=0) \text{ finite})$$

b)

$$2\pi\epsilon_0 rE_r = \lambda_{\text{enclosed}} = \begin{cases} \pi r^2 \rho_1 & r < a \\ \pi a^2 \rho_1 + \pi(r^2 - a^2)\rho_2 & a < r < b \\ \pi a^2 \rho_1 + \pi(b^2 - a^2)\rho_2 & r > b \end{cases}$$

# THE ELECTRIC FIELD

$$E_r = \begin{cases} \frac{\rho_1 r}{2\epsilon_0} & r < a \\ \frac{\rho_1 a^2 + (r^2 - a^2)\rho_2}{2\epsilon_0 r} & a < r < b \\ \frac{\rho_1 a^2 + (b^2 - a^2)\rho_2}{2\epsilon_0 r} & r > b \end{cases}$$

$$c) \quad 2\pi\epsilon_0 r E_r = \begin{cases} \frac{2\pi\rho_0}{a} \int_0^r r'^2 dr' = \frac{2\pi\rho_0 r^3}{3a} & r < a \\ \frac{2\pi\rho_0}{a} \int_0^a r'^2 dr' = \frac{2\pi\rho_0 a^2}{3} & r > a \end{cases}$$

$$E_r = \begin{cases} \frac{\rho_0 r^2}{3\epsilon_0 a} & r < a \\ \frac{\rho_0 a^2}{3\epsilon_0 r} & r > a \end{cases}$$

26. If the hole is filled with uniform charge density  $\rho_0$ , the electric field within the cylinder is

$$\bar{E} = \frac{\rho_0 r \bar{i}_r}{2\epsilon_0} = \frac{\rho_0}{2\epsilon_0} (x \bar{i}_x + y \bar{i}_y) \quad r < R$$

If the hole is filled with uniform charge density  $-\rho_0$ , the electric field within the hole is

$$\bar{E}' = \frac{-\rho_0 r' \bar{i}_{r'}}{2\epsilon_0} = \frac{-\rho_0}{2\epsilon_0} [(x-d) \bar{i}_x + y \bar{i}_y] \quad r' < b$$

where the primed cylindrical coordinate terms are measured with respect to an axis through the center of the hole. The total electric field within the hole is

$$\bar{E}_T = \bar{E} + \bar{E}' = \frac{\rho_0 d}{2\epsilon_0} \bar{i}_x$$

$$27. \quad E_z = - \frac{\partial V}{\partial z} = \frac{\sigma_0}{2\epsilon_0} \rightarrow V = \frac{-\sigma_0 z}{2\epsilon_0} + \text{constant}$$

Each incremental charge element  $\lambda dy$  at coordinate  $y$ , when rotated winds up at position  $z = (\ell - y)$  which requires incremental work



# THE ELECTRIC FIELD

$$dW = \lambda dy [V(z) - V(z=0)]$$

$$= \frac{-\lambda \sigma_0}{2\epsilon_0} (\ell - y) dy$$

The total work required is

$$\begin{aligned} W &= \frac{-\lambda \sigma_0}{2\epsilon_0} \int_0^\ell (\ell - y) dy \\ &= \frac{\lambda \sigma_0}{2\epsilon_0} \frac{(\ell - y)^2}{2} \Big|_0^\ell = \frac{-\lambda \sigma_0 \ell^2}{4\epsilon_0} \end{aligned}$$

$$28. \quad \frac{1}{2} mv^2 + qV = \frac{1}{2} mv^2 + \frac{qQ}{4\pi\epsilon_0 r} = \text{constant} = \frac{1}{2} mv_0^2$$

a) For  $v = 0$  at  $r = R$

$$v_0 \geq \sqrt{\frac{qQ}{2\pi\epsilon_0 Rm}}$$

b) If  $v_0 = \frac{1}{2} \sqrt{\frac{qQ}{2\pi\epsilon_0 Rm}}$ , then  $v = 0$  when

$$\frac{qQ}{4\pi\epsilon_0 r} = \frac{1}{16} \frac{qQ}{\pi\epsilon_0 Rm} \rightarrow r = 4R$$

$$29. \quad a) \quad V = Ax^2 \rightarrow \vec{E} = -\nabla V = -\frac{\partial V}{\partial x} \vec{i}_x = -2Ax\vec{i}_x$$

$$\rho = \epsilon_0 \nabla \cdot \vec{E} = \epsilon_0 \frac{\partial E_x}{\partial x} = -2A\epsilon_0$$

$$b) \quad V = Axyz \rightarrow \vec{E} = -\nabla V = -\frac{\partial V}{\partial x} \vec{i}_x - \frac{\partial V}{\partial y} \vec{i}_y - \frac{\partial V}{\partial z} \vec{i}_z = -A[yz\vec{i}_x + xz\vec{i}_y + xy\vec{i}_z]$$

$$\rho = \epsilon_0 \nabla \cdot \vec{E} = 0$$

$$c) \quad V = Ar^2 \sin\phi + Brz$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r} \vec{i}_r - \frac{1}{r} \frac{\partial V}{\partial \phi} \vec{i}_\phi - \frac{\partial V}{\partial z} \vec{i}_z$$

$$= -[(2Ar \sin\phi + Bz)\vec{i}_r + \text{Arcos}\phi \vec{i}_\phi + Br\vec{i}_z]$$

$$\rho = \epsilon_0 \nabla \cdot \vec{E} = \epsilon_0 \left[ \frac{1}{r} \frac{\partial}{\partial r} (rE_r) + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right]$$

# THE ELECTRIC FIELD

$$= -\epsilon_0 \left[ 3A \sin\phi + \frac{Bz}{r} \right]$$

d)  $V = Ar^2 \sin\theta \cos\phi$

$$\vec{E} = -\nabla V = -\left[ \frac{\partial V}{\partial r} \vec{i}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{i}_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \vec{i}_\phi \right]$$

$$= -A[2r \sin\theta \cos\phi \vec{i}_r + r \cos\theta \cos\phi \vec{i}_\theta - r \sin\phi \vec{i}_\phi]$$

$$\rho = \epsilon_0 \nabla \cdot \vec{E} = \epsilon_0 \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (E_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial E_\phi}{\partial \phi} \right]$$

$$= -\epsilon_0 A \left[ 6 \sin\theta \cos\phi + \frac{\cos\phi}{\sin\theta} \cos 2\theta - \frac{\cos\phi}{\sin\theta} \right]$$

30.  $\nabla \times \vec{E}$  must be zero.

a)  $\vec{E} = ax^2 y^2 \vec{i}_x$

$$\nabla \times \vec{E} = \det \begin{bmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2 y^2 & 0 & 0 \end{bmatrix} = -\vec{i}_z \frac{\partial}{\partial y} (ax^2 y^2) = -2ayx^2 \vec{i}_z \neq 0$$

Not an electric field!

b)  $\vec{E} = a(\vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta) = a\vec{i}_z$  (uniform electric field)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = 0$$

c)  $\vec{E} = a(y\vec{i}_x - x\vec{i}_y)$

$$\nabla \times \vec{E} = \vec{i}_z \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -2a\vec{i}_z \neq 0 \quad (\text{not an electric field})$$

d)  $\vec{E} = \frac{a}{r^2} [\vec{i}_r (1 + \cos\phi) + \sin\phi \vec{i}_\phi]$

$$\nabla \times \vec{E} = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r E_\phi) - \frac{\partial E_r}{\partial \phi} \right] \vec{i}_z$$

$$= \frac{1}{r} \left[ \frac{-\sin\phi}{r^2} + \frac{\sin\phi}{r^2} \right] = 0 \quad (\text{can be an electric field})$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi}$$

# THE ELECTRIC FIELD

$$= \frac{-a}{r^3} (1 + \cos\phi) + \frac{a}{r^3} \cos\phi = \frac{-a}{r^3} \rightarrow \rho = -\frac{\epsilon_0 a}{r^3}$$

31. a)  $E_x = \frac{\sigma_0}{\epsilon_0}$  (between charged sheets)

$$\vec{E} = -\nabla V \rightarrow V = \frac{-\sigma_0 x}{\epsilon_0} + \text{constant}$$

$$\Delta V = V(x=0) - V(x=a) = \frac{\sigma_0 a}{\epsilon_0}$$

b)  $E_r = -\frac{\partial V}{\partial r} = \frac{\lambda_0}{2\pi\epsilon_0 r} \rightarrow V = \frac{-\lambda_0}{2\pi\epsilon_0} \ln r + \text{constant}$

$$\Delta V = V(r=a) - V(r=b) = \frac{\lambda_0}{2\pi\epsilon_0} \ln \frac{b}{a}$$

c)  $E_r = \frac{q_0}{4\pi\epsilon_0 r^2} = -\frac{\partial V}{\partial r} \rightarrow V = \frac{q_0}{4\pi\epsilon_0 r} + \text{constant}$

$$\Delta V = V(r=R_1) - V(r=R_2) = \frac{q_0}{4\pi\epsilon_0} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

32. a)  $dq = \frac{Q}{2\pi R^2} 2\pi R \sin\theta R d\theta = Q \sin\theta d\theta$

$$z' = R \cos\theta; dz' = -R \sin\theta d\theta \rightarrow dq = -\frac{Q}{R} dz'$$

$$\text{radius of hoop } r = \sqrt{R^2 - z'^2}$$

b)  $dV = \frac{dq}{4\pi\epsilon_0 \sqrt{(z-z')^2 + R^2 \sin^2\theta}} = \frac{-Q dz'}{4\pi\epsilon_0 R \sqrt{(z-z')^2 + R^2 - z'^2}}$

$$= \frac{-Q dz'}{4\pi\epsilon_0 R [z^2 - 2zz' + R^2]^{1/2}}$$

c)  $V = \frac{-Q}{4\pi\epsilon_0 R} \int_{z'=R}^0 \frac{dz'}{[z^2 - 2zz' + R^2]^{1/2}}$

$$= \frac{-Q}{4\pi\epsilon_0 R} \frac{2\sqrt{z^2 - 2zz' + R^2}}{(-2z)} \Big|_{z'=R}^0$$

$$= \frac{-Q}{4\pi\epsilon_0 R z} [ |z-R| - \sqrt{z^2 + R^2} ]$$

# THE ELECTRIC FIELD

$$V = \begin{cases} \frac{-Q}{4\pi\epsilon_0 RZ} [z - R - \sqrt{z^2 + R^2}] & z > R \\ \frac{-Q}{4\pi\epsilon_0 RZ} [R - z - \sqrt{z^2 + R^2}] & z < R \end{cases}$$

Check  $z \rightarrow +\infty$

$$V \approx \frac{-Q}{4\pi\epsilon_0 RZ} [z - R - z(1 + \frac{1}{2} \frac{R^2}{z^2})]$$

$$\approx \frac{Q}{4\pi\epsilon_0 z}$$

$z \rightarrow -\infty$

$$V \approx \frac{-Q}{4\pi\epsilon_0 RZ} [R - z + z(1 + \frac{1}{2} \frac{R^2}{z^2})]$$

$$\approx \frac{-Q}{4\pi\epsilon_0 RZ}$$

d)  $z > R$

$$\begin{aligned} E_z &= -\frac{\partial V}{\partial z} = \frac{Q}{4\pi\epsilon_0 R} \frac{\partial}{\partial z} \left\{ 1 - \frac{R}{z} - \frac{\sqrt{z^2 + R^2}}{z} \right\} \\ &= \frac{Q}{4\pi\epsilon_0 R} \left\{ \frac{R}{z^2} + \frac{\sqrt{z^2 + R^2}}{z^2} - \frac{1}{\sqrt{z^2 + R^2}} \right\} \\ &= \frac{Q}{4\pi\epsilon_0 R} \left\{ \frac{R}{z^2} + \frac{R^2}{z^2 \sqrt{z^2 + R^2}} \right\} \\ &= \frac{Q}{4\pi\epsilon_0 z^2} \left\{ 1 + \frac{R}{\sqrt{z^2 + R^2}} \right\} \end{aligned}$$

$z < R$

$$\begin{aligned} E_z &= -\frac{\partial V}{\partial z} = \frac{Q}{4\pi\epsilon_0 R} \frac{\partial}{\partial z} \left[ \frac{R}{z} - 1 - \frac{\sqrt{z^2 + R^2}}{z} \right] \\ &= \frac{Q}{4\pi\epsilon_0 R} \left[ \frac{-R}{z^2} + \frac{\sqrt{z^2 + R^2}}{z^2} - \frac{1}{\sqrt{z^2 + R^2}} \right] \end{aligned}$$

# THE ELECTRIC FIELD

$$= \frac{Q}{4\pi\epsilon_0 R} \left[ \frac{-R}{z^2} + \frac{R^2}{z^2 \sqrt{z^2 + R^2}} \right]$$

$$= \frac{Q}{4\pi\epsilon_0 z^2} \left[ -1 + \frac{R}{\sqrt{z^2 + R^2}} \right]$$

Check:

$$\sigma_f = \epsilon_0 [E_z(z=R_+) - E_z(z=R_-)] = \frac{Q}{2\pi R^2}$$

e)  $\sigma_o = \frac{Q}{2\pi R^2} \rightarrow \rho_o dr$

$$dV = \begin{cases} -\frac{\rho_o r dr}{2\epsilon_o z} [z - r - \sqrt{z^2 + r^2}] & z > r \\ -\frac{\rho_o r dr}{2\epsilon_o z} [r - z - \sqrt{z^2 + r^2}] & z < r \end{cases}$$

$z \geq R$

$$V = \frac{-\rho_o}{2\epsilon_o z} \int_{r=0}^R [rz - r^2 - r\sqrt{z^2 + r^2}] dr$$

$$= \frac{-\rho_o}{2\epsilon_o z} \left[ \frac{R^2 z}{2} - \frac{R^3}{3} - \frac{1}{3} [R^2 + z^2]^{3/2} + \frac{z^3}{3} \right]$$

$z \leq 0$

$$V = \frac{-\rho_o}{2\epsilon_o z} \int_{r=0}^R [r^2 - zr - r\sqrt{z^2 + r^2}] dr$$

$$= \frac{-\rho_o}{2\epsilon_o z} \left[ \frac{R^3}{3} - \frac{zR^2}{2} - \frac{1}{3} [R^2 + z^2]^{3/2} + \frac{z^3}{3} \right]$$

$0 \leq z \leq R$

$$V = \frac{-\rho_o}{2\epsilon_o z} \left[ \int_0^z [rz - r^2 - r\sqrt{z^2 + r^2}] dr + \int_z^R [r^2 - zr - r\sqrt{z^2 + r^2}] dr \right]$$

$$= \frac{-\rho_o}{2\epsilon_o z} \left[ \frac{z^3}{2} - \frac{z^3}{3} - \frac{1}{3} [2z^2]^{3/2} + \frac{z^3}{3} \right.$$

$$\left. + \frac{R^3}{3} - \frac{z^3}{2} (R^2 - z^2) - \frac{1}{3} (R^2 + z^2)^{3/2} + \frac{1}{3} (2z^2)^{3/2} \right]$$

# THE ELECTRIC FIELD

$$E_z = -\frac{\partial V}{\partial z} = \begin{cases} -\frac{\rho_o}{2\epsilon_o z} \left[ \frac{R^3}{3} - \frac{R^2 z}{2} - \frac{1}{3} (R^2 + z^2)^{3/2} + \frac{2}{3} z^3 \right] & z \geq R \\ \frac{\rho_o}{2\epsilon_o} \left[ \frac{R^3}{3z^2} + \frac{2}{3} z - \frac{1}{3} [R^2 + z^2]^{1/2} \left[ 2 - \frac{R^2}{z^2} \right] \right] & z \leq 0 \\ \frac{\rho_o}{2\epsilon_o} \left[ -\frac{R^3}{3z^2} + \frac{2z}{3} - \frac{1}{3} [R^2 + z^2]^{1/2} \left[ 2 - \frac{R^2}{z^2} \right] \right] & 0 \leq z \leq R \end{cases}$$

33.  $V = \frac{1}{4\pi\epsilon_o} \left[ \frac{q_1}{r_1} + \frac{q_2}{r_2} \right]$

$$r_1 = [r^2 + (\frac{a}{2})^2 - \arccos\theta]^{1/2}; \quad r_2 = [r^2 + (\frac{a}{2})^2 + \arccos\theta]^{1/2}$$

a)  $V = \frac{1}{4\pi\epsilon_o} \left\{ \frac{q_1}{[r^2 + (\frac{a}{2})^2 - \arccos\theta]^{1/2}} + \frac{q_2}{[r^2 + (\frac{a}{2})^2 + \arccos\theta]^{1/2}} \right\}$

b)  $\vec{E} = -\nabla V = -\left[ \frac{\partial V}{\partial r} \vec{i}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{i}_\theta \right]$

$$= -\frac{1}{4\pi\epsilon_o} \left\{ \frac{q_1 \left[ -\frac{1}{2} [(2r - \arccos\theta) \vec{i}_r + \arccos\theta \vec{i}_\theta] \right]}{[r^2 + (\frac{a}{2})^2 - \arccos\theta]^{3/2}} + \frac{q_2 \left[ -\frac{1}{2} [(2r + \arccos\theta) \vec{i}_r - \arccos\theta \vec{i}_\theta] \right]}{[r^2 + (\frac{a}{2})^2 + \arccos\theta]^{3/2}} \right\}$$

$$= \frac{1}{8\pi\epsilon_o} \left\{ \frac{q_1 [(2r - \arccos\theta) \vec{i}_r + \arccos\theta \vec{i}_\theta]}{[r^2 + (\frac{a}{2})^2 - \arccos\theta]^{3/2}} + \frac{q_2 [(2r + \arccos\theta) \vec{i}_r - \arccos\theta \vec{i}_\theta]}{[r^2 + (\frac{a}{2})^2 + \arccos\theta]^{3/2}} \right\}$$

c)  $\lim r \gg a, q_1 = -q_2 \equiv q_o$

$$[r^2 + (\frac{a}{2})^2 - \arccos\theta] \approx r^2 \left[ 1 + \frac{a^2}{4r^2} - \frac{a}{r} \cos\theta \right] \approx r^2 (1 - \frac{a}{r} \cos\theta)$$

$$r_1 \approx r (1 - \frac{1}{2} \frac{a}{r} \cos\theta); \quad r_2 \approx r (1 + \frac{1}{2} \frac{a}{r} \cos\theta)$$

$$\begin{aligned} V &\approx \frac{q_o}{4\pi\epsilon_o r} \left\{ (1 + \frac{1}{2} \frac{a}{r} \cos\theta) - (1 - \frac{1}{2} \frac{a}{r} \cos\theta) \right\} \\ &= \frac{q_o a}{4\pi\epsilon_o r^2} \cos\theta \end{aligned}$$

# THE ELECTRIC FIELD

$$\vec{E} \approx \frac{q_o}{8\pi\epsilon_o} \left\{ \frac{(1 + \frac{3a}{2r} \cos\theta)}{r^3} [(2r - a\cos\theta)\vec{i}_r + a\sin\theta\vec{i}_\theta] - \frac{(1 - \frac{3a}{2r} \cos\theta)}{r^3} [(2r + a\cos\theta)\vec{i}_r - a\sin\theta\vec{i}_\theta] \right\}$$

$$\approx \frac{q_o}{8\pi\epsilon_o r^3} \{4a\cos\theta\vec{i}_r + 2a\sin\theta\vec{i}_\theta\}$$

$$\approx \frac{q_o a}{4\pi\epsilon_o r^3} [2\cos\theta\vec{i}_r + \sin\theta\vec{i}_\theta]$$

$$d) \frac{dr}{r d\theta} = \frac{E}{E_\theta} = 2\cot\theta \rightarrow \int \frac{dr}{r} = \int 2\cot\theta d\theta \rightarrow \ln r = 2\ln\sin\theta + \text{constant} \rightarrow r = r_o \sin^2\theta$$

$$34. a) V_1 = \frac{1}{4\pi\epsilon_o} \left( \frac{q_2}{r_{12}} + \frac{q_3}{r_{13}} \right)$$

$$V_2 = \frac{1}{4\pi\epsilon_o} \left( \frac{q_1}{r_{12}} + \frac{q_3}{r_{23}} \right)$$

$$V_3 = \frac{1}{4\pi\epsilon_o} \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$$

$$b) V'_1 = \frac{1}{4\pi\epsilon_o} \left( \frac{q'_2}{r_{12}} + \frac{q'_3}{r_{13}} \right)$$

$$V'_2 = \frac{1}{4\pi\epsilon_o} \left( \frac{q'_1}{r_{12}} + \frac{q'_3}{r_{23}} \right)$$

$$V'_3 = \frac{1}{4\pi\epsilon_o} \left( \frac{q'_1}{r_{13}} + \frac{q'_2}{r_{23}} \right)$$

$$q'_1 V_1 + q'_2 V_2 + q'_3 V_3 = q_1 V'_1 + q_2 V'_2 + q_3 V'_3$$

- c) By breaking each conductor into many incremental point charges, because each are at the same potential (the conductor is an equipotential surface) the potential may be taken outside the summation

$$\sum_{\text{on conductor}} (q_i V'_i - q'_i V_i) = V'_i \sum q_i - V_i \sum q'_i$$

Total charge on conductor

# THE ELECTRIC FIELD

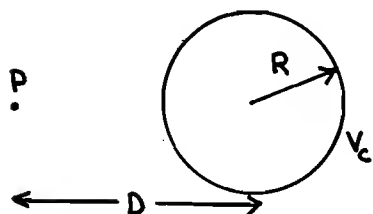
d) Let  $q_1 = q$ ,  $q_2 = q_c$ ,  $V_2 = 0$

$$q'_1 = 0, V'_1 = V_p, V'_2 = V_c$$

$$\cancel{q'_1} \overset{0}{V_1} + \cancel{q'_2} \overset{0}{V_2} = \underbrace{q_1 V'_1}_{qV_p} + \underbrace{q_2 V'_2}_{q_c V_c}$$

$$qV_p + q_c V_c = 0 \rightarrow q_c = -\frac{qV_p}{V_c}$$

e)



$$V_p = \frac{V_c R}{D}, \quad q_c = -\frac{qR}{D}$$

f)  $V = C_1 \ln r/r_o \rightarrow V_c = C_1 \ln \frac{a}{r_o} \rightarrow C_1 = \frac{V_c}{\ln \frac{a}{r_o}} \quad (r_o \rightarrow \infty)$

$$V_p = \frac{V_c}{\ln \frac{a}{r_o}} \ln \frac{D}{r_o} \rightarrow \lambda_c = -\frac{\lambda V_p}{V_c} = -\lambda \frac{\ln D/r_o}{\ln a/r_o}$$

$$\lim_{r_o \rightarrow \infty} \lambda_c = -\lambda$$

g)  $q_1 = q$ ,  $q_2 = q(y=0)$ ,  $q_3 = q(y=d)$ ,  $V_2 = 0$ ,  $V_3 = 0$

$$q'_1 = 0, V'_2 = V_o, V'_3 = 0$$

$$\cancel{q'_1} \overset{0}{V_1} + \cancel{q'_2} \overset{0}{V_2} + \cancel{q'_3} \overset{0}{V_3} = \underbrace{q'_1 V'_1}_{qV'_1} + \underbrace{q'_2 V'_2}_{q(y=0)V_o} + \underbrace{q'_3 V'_3}_{0}$$

$$qV'_1 + q(y=0)V_o = 0$$

$$V'_1 = V_o \left(1 - \frac{a}{d}\right)$$

$$q(y=0) = -\frac{qV'_1}{V_o} = -q \left(1 - \frac{a}{d}\right) = -\frac{qb}{d}$$

$$q(y=d) = -q - q(y=0) = -q \left[1 - \frac{b}{d}\right] = -\frac{qa}{d}$$



# THE ELECTRIC FIELD

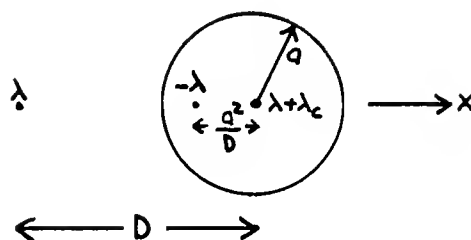
$$h) \quad q(y=0) = -\frac{q}{d} (d - x)$$

$$q(y=d) = -\frac{qx}{d}$$

$$i = \frac{dq(y=0)}{dt} = -\frac{dq(y=d)}{dt} = \frac{q}{d} \frac{dx}{dt} = \frac{qv_o}{d}$$

## Section 2.6

$$35. \quad f_x = \frac{\lambda}{2\pi\epsilon_0} \left[ \frac{-\lambda}{D - \frac{a^2}{D}} + \frac{(\lambda + \lambda_c)}{D} \right]$$



$$36. \quad a) \quad dE_y = \frac{\sigma_o dy'}{2\pi\epsilon_0 (y - y')} \quad y > y'$$

$$y > d$$

$$E_y = \frac{\sigma_o}{2\pi\epsilon_0} \int_{y'=0}^d \frac{dy'}{(y - y')}$$

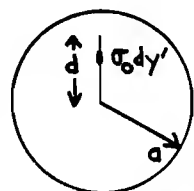
$$= \frac{-\sigma_o}{2\pi\epsilon_0} \ln(y - y') \Big|_{y'=0}^d = \frac{-\sigma_o}{2\pi\epsilon_0} \ln(1 - \frac{d}{y})$$

$$0 < y < d$$

$$E_y = \frac{\sigma_o}{2\pi\epsilon_0} \left[ \int_0^y \frac{dy'}{(y - y')} - \int_y^d \frac{dy'}{(y' - y)} \right]$$

$$= \frac{\sigma_o}{2\pi\epsilon_0} \ln \frac{y}{d - y}$$

$$b)$$



$$y < 0$$

$$E_y = -\frac{\sigma_o}{2\pi\epsilon_0} \int_{y'=0}^d \frac{dy'}{(y' - y)}$$

$$= -\frac{\sigma_o}{2\pi\epsilon_0} \ln \left( 1 - \frac{d}{y} \right)$$

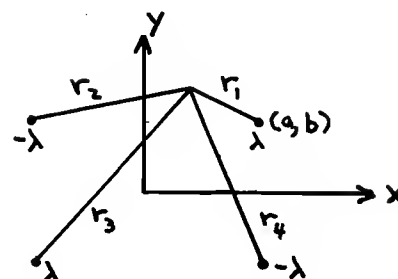
$$c) \quad d\vec{F} = \vec{i}_y \sigma_o dy' E_y \left( y = \frac{a^2}{y'} \right)$$

$$= \vec{i}_y \sigma_o dy' \left( \frac{-\sigma_o}{2\pi\epsilon_0} \right) \ln \left( 1 - \frac{dy'}{a^2} \right)$$

$$\vec{F} = \frac{-\sigma_o^2}{2\pi\epsilon_0} \vec{i}_y \int_{y'=0}^d \ln \left( 1 - \frac{dy'}{a^2} \right) dy'$$

$$= \frac{-\sigma_o^2 a^2}{2\pi\epsilon_0 d} \left( 1 - \frac{dy'}{a^2} \right) \left[ \ln \left( 1 - \frac{dy'}{a^2} \right) - 1 \right] \Big|_{y'=0}^d \vec{i}_y$$

$$= \frac{\sigma_o^2 a^2}{2\pi\epsilon_o d} \left\{ \left(1 - \frac{d^2}{a^2}\right) \left[\ln\left(1 - \frac{d^2}{a^2}\right) - 1\right] + 1 \right\} \bar{i}_y$$



$$37. \quad a) \quad V = \frac{-\lambda}{2\pi\epsilon_o} [\ln r_1 - \ln r_2 + \ln r_3 - \ln r_4]$$

$$r_1 = [(y - b)^2 + (x - a)^2]^{1/2}; \quad r_2 = [(y - b)^2 + (x + a)^2]^{1/2}$$

$$r_3 = [(y + b)^2 + (x + a)^2]^{1/2}; \quad r_4 = [(y + b)^2 + (x - a)^2]^{1/2}$$

$$\begin{aligned} \bar{E} = -\nabla V = \frac{\lambda}{2\pi\epsilon_o} & \left\{ \frac{[(y - b)\bar{i}_y + (x - a)\bar{i}_x]}{r_1^2} - \frac{[(y - b)\bar{i}_y + (x + a)\bar{i}_x]}{r_2^2} \right. \\ & \left. + \frac{[(y + b)\bar{i}_y + (x + a)\bar{i}_x]}{r_3^2} - \frac{[(y + b)\bar{i}_y + (x - a)\bar{i}_x]}{r_4^2} \right\} \end{aligned}$$

$$E_x(y=0) = 0, \quad E_y(x=0) = 0$$

$$b) \quad \text{At } x = a, \quad y = b$$

$$r_1 = 0, \quad r_2 = 2a, \quad r_3 = \sqrt{(2a)^2 + (2b)^2}, \quad r_4 = 2b$$

$$\begin{aligned} \bar{F} &= \frac{\lambda^2}{2\pi\epsilon_o} \left[ \frac{-\bar{i}_x}{2a} - \frac{\bar{i}_y}{2b} + \frac{(a\bar{i}_x + b\bar{i}_y)}{2(a^2 + b^2)} \right] \\ &= \frac{\lambda^2}{4\pi\epsilon_o} \left[ -\bar{i}_x \left( \frac{1}{a} - \frac{a}{a^2 + b^2} \right) - \bar{i}_y \left( \frac{1}{b} - \frac{b}{a^2 + b^2} \right) \right] \\ &= \frac{-\lambda^2}{4\pi\epsilon_o (a^2 + b^2)ab} \{ b^3 \bar{i}_x + a^3 \bar{i}_y \} \end{aligned}$$

$$c) \quad \sigma_f(x=0) = \epsilon_o E_x(x=0) = \frac{\lambda}{2\pi} \left( \frac{2a}{r_2} - \frac{2a}{r_1} \right) = \frac{\lambda a}{\pi} \left( \frac{1}{[(y+b)^2 + a^2]} - \frac{1}{[(y-b)^2 + a^2]} \right)$$

$$\sigma_f(y=0) = \epsilon_o E_y(y=0) = \frac{\lambda}{2\pi} \left( \frac{-2b}{r_1} + \frac{2b}{r_2} \right) = \frac{\lambda b}{\pi} \left( \frac{1}{[(x-a)^2 + b^2]} - \frac{1}{[(x+a)^2 + b^2]} \right)$$

$$\begin{aligned}
 \lambda(x=0) &= \int_0^{\infty} \sigma_f(x=0) dy \\
 &= \frac{\lambda a}{\pi} \int_0^{\infty} \left[ \frac{1}{[(y+b)^2 + a^2]} - \frac{1}{[(y-b)^2 + a^2]} \right] dy \\
 &= \frac{\lambda a}{\pi} \left\{ \frac{1}{a} \tan^{-1} \frac{y+b}{a} - \frac{1}{a} \tan^{-1} \frac{y-b}{a} \right\} \Big|_0^{\infty} \\
 &= \frac{\lambda}{\pi} \left\{ \frac{\pi}{2} - \tan^{-1} \frac{b}{a} - \frac{\pi}{2} - \tan^{-1} \frac{b}{a} \right\} \\
 &= -\frac{2\lambda}{\pi} \tan^{-1} \frac{b}{a}
 \end{aligned}$$

$$\lambda(y=0) = -\frac{2\lambda}{\pi} \tan^{-1} \frac{a}{b}$$

$$\begin{aligned}
 \lambda(x=0) + \lambda(y=0) &= -\frac{2\lambda}{\pi} \left[ \tan^{-1} \frac{a}{b} + \tan^{-1} \frac{b}{a} \right] \\
 &\quad \underbrace{\hspace{1.5cm}}_{\pi/2} \\
 &= -\lambda
 \end{aligned}$$

d) Use single image charge at (a, -b)

$$\begin{aligned}
 \bar{F} &= \frac{\lambda^2}{2\pi\epsilon_0} \left[ +\frac{\bar{i}_y}{2b} - \frac{\bar{i}_x}{2a} + \frac{(a\bar{i}_x - b\bar{i}_y)}{2(a^2 + b^2)} \right] \\
 &= \frac{\lambda^2}{4\pi\epsilon_0} \left[ -\bar{i}_x \left( \frac{1}{a} - \frac{a}{(a^2 + b^2)} \right) + \bar{i}_y \left( \frac{1}{b} - \frac{b}{(a^2 + b^2)} \right) \right] \\
 &= \frac{\lambda^2}{4\pi\epsilon_0 ab(a^2 + b^2)} [-b^3 \bar{i}_x + a^3 \bar{i}_y]
 \end{aligned}$$

$$e) \quad V = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right]$$

$$r_1 = [(x-a)^2 + (y-b)^2 + z^2]^{1/2}; \quad r_2 = [(x+a)^2 + (y-b)^2 + z^2]^{1/2}$$

$$r_3 = [(x+a)^2 + (y+b)^2 + z^2]^{1/2}; \quad r_4 = [(x-a)^2 + (y+b)^2 + z^2]^{1/2}$$

# THE ELECTRIC FIELD

$$\vec{E} = -\nabla V = \frac{q}{4\pi\epsilon_0} \left[ \frac{[(x-a)\vec{i}_x + (y-b)\vec{i}_y + z\vec{i}_z]}{r_1^3} - \frac{[(x+a)\vec{i}_x + (y-b)\vec{i}_y + z\vec{i}_z]}{r_2^3} + \frac{[(x+a)\vec{i}_x + (y+b)\vec{i}_y + z\vec{i}_z]}{r_3^3} - \frac{[(x-a)\vec{i}_x + (y+b)\vec{i}_y + z\vec{i}_z]}{r_4^3} \right]$$

$$\vec{F} = \frac{q^2}{16\pi\epsilon_0} \left\{ a \left[ \frac{1}{(a^2 + b^2)^{3/2}} - \frac{1}{a^3} \right] \vec{i}_x + b \left[ \frac{1}{(a^2 + b^2)^{3/2}} - \frac{1}{b^3} \right] \vec{i}_y \right\}$$

$$\sigma_f(x=0) = \epsilon_0 E_x(x=0), \quad \sigma_f(y=0) = \epsilon_0 E_y(y=0)$$

$$q(x=0) = \int_{y=0}^{\infty} \int_{z=-\infty}^{+\infty} \sigma_f(x=0) dy dz = \frac{-2q}{\pi} \tan^{-1} \frac{b}{a}$$

$$q(y=0) = \int_{x=0}^{\infty} \int_{z=-\infty}^{+\infty} \sigma_f(y=0) dx dz = \frac{-2q}{\pi} \tan^{-1} \frac{a}{b}$$

If charges are outside conducting corner

$$\vec{F} = \frac{q^2}{16\pi\epsilon_0} \left\{ -a \left[ \frac{1}{a^3} - \frac{1}{(a^2 + b^2)^{3/2}} \right] \vec{i}_x + b \left[ \frac{1}{b^3} - \frac{1}{(a^2 + b^2)^{3/2}} \right] \vec{i}_y \right\}$$

## Section 2.7

$$38. F_x = qE_o - \frac{q^2}{4\pi\epsilon_0 (2x)^2}$$

$$= qE_o - \frac{q^2}{16\pi\epsilon_0 x^2}$$

$$a) F_x = 0 \rightarrow x_o^2 = \frac{q}{16\pi\epsilon_0 E_o} \rightarrow x_o = \sqrt{\frac{q}{16\pi\epsilon_0 E_o}}$$

$$b) V(x) = -E_o x - \frac{q}{16\pi\epsilon_0 x}$$

$$V(x_o) = -E_o x_o - \frac{q}{16\pi\epsilon_0 x_o} = -\frac{1}{2} \sqrt{\frac{qE_o}{\pi\epsilon_0}}$$

$$V\left(\frac{x_o}{2}\right) = -\frac{5}{8} \sqrt{\frac{qE_o}{\pi\epsilon_0}}$$

# THE ELECTRIC FIELD

$$\frac{1}{2} mv^2 + qV(x) = \text{constant}$$

For  $v = 0$  at  $x = x_0$

$$\frac{1}{2} mv_0^2 + qV\left(\frac{x_0}{2}\right) = 0 + qV(x_0)$$

$$\begin{aligned} \frac{1}{2} mv_0^2 &> q[V(x_0) - V\left(\frac{x_0}{2}\right)] \\ &> q \sqrt{\frac{qE_0}{\pi\epsilon_0}} \left[-\frac{1}{2} + \frac{5}{8}\right] \\ &> \frac{q}{8} \sqrt{\frac{qE_0}{\pi\epsilon_0}} \end{aligned}$$

$$v_0 > \frac{1}{2} \sqrt{\frac{q}{m} \left[\frac{qE_0}{\pi\epsilon_0}\right]^{1/4}}$$

c) If  $E_0 = 0$ ,  $V = \frac{-q}{16\pi\epsilon_0 x}$

$$W = q[V(\infty) - V(d)] = \frac{q^2}{16\pi\epsilon_0 d}$$

39. a)  $Q_{\text{ind}} = -\frac{R_1}{R_2} Q$

b)  $E_r = \begin{cases} \frac{Q_{\text{ind}}}{4\pi\epsilon_0 r^2} = -\frac{R_1}{R_2} \frac{Q}{4\pi\epsilon_0 r^2} \\ \frac{Q + Q_{\text{ind}}}{4\pi\epsilon_0 r^2} = \frac{(1 - \frac{R_1}{R_2})Q}{4\pi\epsilon_0 r^2} \end{cases}$

$$R_1 < r < R_2$$

$$r > R_2$$

$$E_r = -\frac{\partial V}{\partial r} \rightarrow V = \begin{cases} \frac{-R_1}{R_2} \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{R_1}\right) \\ \frac{(1 - \frac{R_1}{R_2})Q}{4\pi\epsilon_0 r} \end{cases}$$

$$R_1 < r < R_2$$

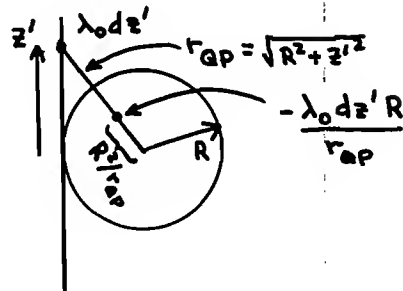
$$r > R_2$$

# THE ELECTRIC FIELD

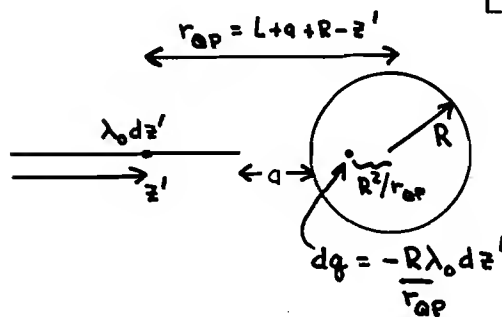
40. Field inside the sphere due to  $V_0$  is zero. Only field outside the sphere affected by raising the potential.

$$F_x = \frac{q(-q \frac{R}{D})}{4\pi\epsilon_0 \left[ \frac{R^2}{D} - D \right]^2} = \frac{-q^2 RD}{4\pi\epsilon_0 [R^2 - D^2]^2}$$

41. a)

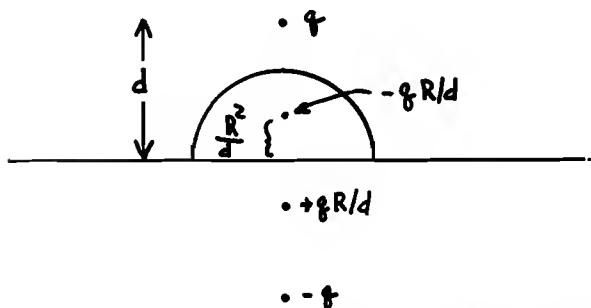


$$\begin{aligned} b) \quad q_T &= \int_{-L/2}^{+L/2} \frac{-\lambda_0 R}{\sqrt{R^2 + z'^2}} dz' = -\lambda_0 R \ln \left[ z' + \sqrt{R^2 + z'^2} \right] \Big|_{-L/2}^{+L/2} \\ &= -\lambda_0 R \ln \left[ \frac{\frac{L}{2} + \sqrt{R^2 + \frac{L^2}{4}}}{-\frac{L}{2} + \sqrt{R^2 + \frac{L^2}{4}}} \right] \end{aligned}$$



$$\begin{aligned} q_T &= \int_0^L \frac{-R\lambda_0 dz'}{L + a + R - z'} = R\lambda_0 \ln(L + a + R - z') \Big|_0^L \\ &= R\lambda_0 \ln \frac{R + a}{R + a + L} \end{aligned}$$

- 42.



# THE ELECTRIC FIELD

$$\begin{aligned}
 \text{a) } F_z &= q \left[ \frac{-qR/d}{4\pi\epsilon_0 (d - \frac{R^2}{d})^2} + \frac{qR/d}{4\pi\epsilon_0 (d + \frac{R^2}{d})^2} - \frac{q}{4\pi\epsilon_0 (2d)^2} \right] \\
 &= \frac{-q^2 R}{4\pi\epsilon_0} \left[ \frac{d}{(d^2 - R^2)^2} - \frac{d}{(d^2 + R^2)^2} + \frac{1}{4Rd^2} \right] \\
 &= \frac{-q^2}{4\pi\epsilon_0} \left[ \frac{4R^3 d^3}{(d^4 - R^4)^2} + \frac{1}{4d^2} \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } V &= \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{[x^2 + y^2 + (z - d)^2]^{1/2}} - \frac{R/d}{[x^2 + y^2 + (z - \frac{R^2}{d})^2]^{1/2}} \right. \\
 &\quad \left. + \frac{R/d}{[x^2 + y^2 + (z + \frac{R^2}{d})^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z + d)^2]^{1/2}} \right\} \\
 \vec{E} &= -\nabla V = \frac{q}{4\pi\epsilon_0} \left\{ \frac{(x\vec{i}_x + y\vec{i}_y + (z-d)\vec{i}_z)}{[x^2 + y^2 + (z-d)^2]^{3/2}} - \frac{(R/d)(x\vec{i}_x + y\vec{i}_y + (z - \frac{R^2}{d})\vec{i}_z)}{[x^2 + y^2 + (z - \frac{R^2}{d})^2]^{3/2}} \right. \\
 &\quad \left. + \frac{(R/d)(x\vec{i}_x + y\vec{i}_y + (z + \frac{R^2}{d})\vec{i}_z)}{[x^2 + y^2 + (z + \frac{R^2}{d})^2]^{3/2}} - \frac{(x\vec{i}_x + y\vec{i}_y + (z+d)\vec{i}_z)}{[x^2 + y^2 + (z+d)^2]^{3/2}} \right\}
 \end{aligned}$$

At  $z = 0$

$$\begin{aligned}
 \vec{E} &= \vec{i}_z \frac{q}{4\pi\epsilon_0} \left\{ \frac{-2d}{[x^2 + y^2 + d^2]^{3/2}} + \frac{2R^3/d^2}{[x^2 + y^2 + (\frac{R^2}{d})^2]^{3/2}} \right\} \\
 &= \vec{i}_z \frac{qd}{2\pi\epsilon_0} \left\{ \frac{-1}{[r^2 + d^2]^{3/2}} + \frac{R^3/d^3}{[r^2 + (\frac{R^2}{d})^2]^{3/2}} \right\}
 \end{aligned}$$

$$\sigma_f(z=0) = \epsilon_0 E_z(z=0)$$

$$q_T(z=0) = \int_{r=R}^{\infty} 2\pi\sigma_f(z=0)rdr$$

# THE ELECTRIC FIELD

$$\begin{aligned}
 &= qd \int_{r=R}^{\infty} \left[ \frac{-r}{[r^2 + d^2]^{3/2}} + \frac{r \left(\frac{R}{d}\right)^3}{[r^2 + \left(\frac{R^2}{d}\right)^2]^{3/2}} \right] dr \\
 &= qd \left[ \frac{1}{\sqrt{r^2 + d^2}} - \frac{(R/d)^3}{\sqrt{r^2 + \left(\frac{R^2}{d}\right)^2}} \right] \Big|_{r=R}^{\infty} \\
 &= qd \left[ \frac{(R/d)^3}{\sqrt{R^2 + (R^2/d)^2}} - \frac{1}{\sqrt{R^2 + d^2}} \right] \\
 &= q \frac{\left(\frac{R^2}{d^2} - 1\right)}{\sqrt{1 + \frac{R^2}{d^2}}}
 \end{aligned}$$

The easiest way to find the total charge induced on the hemisphere is to realize that the total induced charge is  $-q$

$$q_T(r=R) + q_T(z=0) = -q \rightarrow q_T(r=R) = -q \left[ 1 + \frac{\frac{R^2}{d^2} - 1}{\sqrt{1 + \frac{R^2}{d^2}}} \right]$$

To verify this we convert the Cartesian coordinates to spherical coordinates

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$\bar{i}_x = \sin \theta \cos \phi \bar{i}_r + \cos \theta \cos \phi \bar{i}_\theta - \sin \phi \bar{i}_\phi$$

$$\bar{i}_y = \sin \theta \sin \phi \bar{i}_r + \cos \theta \sin \phi \bar{i}_\theta + \cos \phi \bar{i}_\phi$$

$$\bar{i}_z = \cos \theta \bar{i}_r - \sin \theta \bar{i}_\theta$$

At  $r = R$  we have

$$x^2 + y^2 + z^2 = R^2$$

$$x^2 + y^2 + (z \pm d)^2 = R^2 + d^2 \pm 2Rd \cos \theta$$

$$x^2 + y^2 + \left(z \pm \frac{R^2}{d}\right)^2 = R^2 \left[ 1 + \frac{R^2}{d^2} \pm \frac{2R}{d} \cos \theta \right]$$

The radial component of electric field at  $r = R$  is then



# THE ELECTRIC FIELD

$$E_r(r=R) = \frac{q}{4\pi\epsilon_o} \left\{ \frac{(R - d\cos\theta)}{[R^2 + d^2 - 2Rd\cos\theta]^{3/2}} - \frac{(1 - \frac{R}{d}\cos\theta)}{Rd[1 + \frac{R^2}{d^2} - \frac{2R}{d}\cos\theta]^{3/2}} \right. \\ \left. + \frac{(1 + \frac{R}{d}\cos\theta)}{Rd[1 + \frac{R^2}{d^2} + \frac{2R}{d}\cos\theta]^{3/2}} - \frac{(R + d\cos\theta)}{[R^2 + d^2 + 2Rd\cos\theta]^{3/2}} \right\}$$

The surface charge density is then

$$\sigma_f(r=R) = \epsilon_o E_r(r=R)$$

with total charge

$$q_T(r=R) = 2\pi R^2 \int_{\theta=0}^{\pi/2} \sigma_f(r=R) \sin\theta d\theta = -2\pi R^2 \int_{u=1}^0 \sigma_f(r=R) du \quad (u = \cos\theta)$$

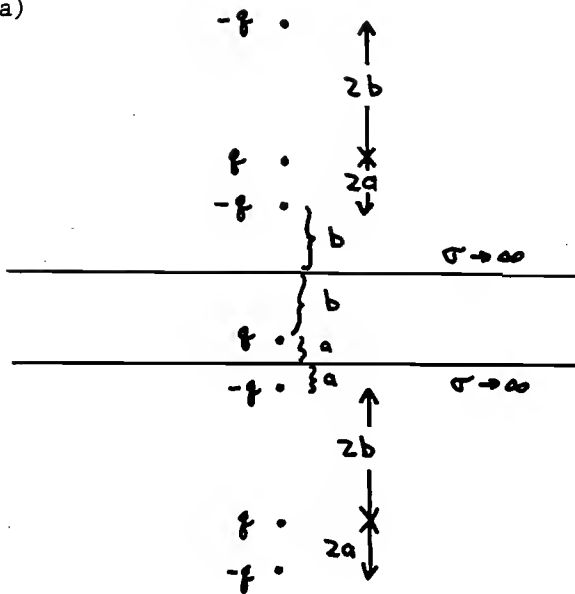
$$= \frac{qR^2}{2} \left\{ \frac{-(Ru - d)}{R^2[R^2 + d^2 - 2Rdu]^{1/2}} + \frac{(Ru + d)}{R^2[R^2 + d^2 + 2Rdu]^{1/2}} \right. \\ \left. + \frac{(u - \frac{R}{d})}{Rd[1 + \frac{R^2}{d^2} - \frac{2Ru}{d}]^{1/2}} - \frac{(u + \frac{R}{d})}{Rd[1 + \frac{R^2}{d^2} + \frac{2Ru}{d}]^{1/2}} \right\} \Big|_{u=1}^0 \\ = \frac{q}{2} \left\{ \frac{2d}{[R^2 + d^2]^{1/2}} - \frac{2R^2}{d[R^2 + d^2]^{1/2}} + \frac{(R - d)}{(d - R)} - \frac{R + d}{(R + d)} \right. \\ \left. - \frac{R(1 - \frac{R}{d})}{d(1 - \frac{R}{d})} + \frac{R(1 + \frac{R}{d})}{d(1 + \frac{R}{d})} \right\}$$

$$q_T(r=R) = q \left\{ \frac{d - \frac{R^2}{d}}{[R^2 + d^2]^{1/2}} - 1 \right\} \\ = -q \left\{ 1 + \frac{\frac{R^2}{d^2} - 1}{\sqrt{1 + \frac{R^2}{d^2}}} \right\}$$

which agrees with our earlier result.

# THE ELECTRIC FIELD

43. a)



b) The total charge on each conductor is

$$q_T = -q[1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \dots] \text{ non-convergent series}$$

c)  $q_{n+1} = -\frac{R_1}{b'_n} q'_n, \quad q'_{n+1} = -\frac{R_2}{b_n} q_n$

$$b_{n+1} = \frac{R_1^2}{b'_n}, \quad b'_{n+1} = \frac{R_2^2}{b_n}$$

d)  $q_{n+1} = -\frac{R_1}{b'_n} q'_n = -\frac{R_1}{b'_n} \left( -\frac{R_2}{b_{n-1}} q_{n-1} \right) = \underbrace{\frac{R_1 R_2}{b'_n b_{n-1}}}_{R_2^2} q_{n-1}$

$$q_{n+1} = \frac{R_1}{R_2} q_{n-1}$$

$$b_{n+1} = \frac{R_1^2}{b'_n} = \frac{R_1^2}{R_2^2} b_{n-1}$$

e)  $q_n = A\lambda^n$

$$A\left[\lambda^{n+1} - \frac{R_1}{R_2} \lambda^{n-1}\right] = A\lambda^{n-1}\left[\lambda^2 - \frac{R_1}{R_2}\right] = 0 \rightarrow \lambda = \pm \left(\frac{R_1}{R_2}\right)^{1/2}$$

$$b_n = B\alpha^n$$

THE ELECTRIC FIELD

$$0 = B[\alpha^{n+1} - (\frac{R_1}{R_2})^2 \alpha^{n-1}] = B\alpha^{n-1}[\alpha^2 - (\frac{R_1}{R_2})^2] \rightarrow \alpha = \pm \frac{R_1}{R_2}$$

$$f) \quad q_n = A_1 (\frac{R_1}{R_2})^{n/2} + A_2 (-1)^n (\frac{R_1}{R_2})^{n/2} = (\frac{R_1}{R_2})^{n/2} [A_1 + (-1)^n A_2]$$

$$b_n = B_1 (\frac{R_1}{R_2})^n + B_2 (-1)^n (\frac{R_1}{R_2})^n = (\frac{R_1}{R_2})^n [B_1 + (-1)^n B_2]$$

$$\left. \begin{aligned} q_1 &= -\frac{qR_1}{R_0} = (A_1 - A_2) (\frac{R_1}{R_2})^{1/2} \\ q_2 &= \frac{qR_1}{R_2} = (A_1 + A_2) (\frac{R_1}{R_2}) \end{aligned} \right\} \rightarrow \begin{aligned} A_1 - A_2 &= -q (\frac{R_1 R_2}{R_0^2})^{1/2} \\ A_1 + A_2 &= q \end{aligned}$$

$$q_n = \begin{cases} q (\frac{R_1}{R_2})^{n/2} & n \text{ even} \\ -q \frac{R_1}{R_0} (\frac{R_1}{R_2})^{(n-1)/2} & n \text{ odd} \end{cases}$$

$$b_1 = \frac{R_1^2}{R_0} = (B_1 - B_2) \frac{R_1}{R_2} \rightarrow B_1 - B_2 = \frac{R_1 R_2}{R_0}$$

$$b_2 = \frac{R_1^2 R_0}{R_2^2} = (B_1 + B_2) (\frac{R_1}{R_2})^2 \rightarrow B_1 + B_2 = R_0$$

$$b_n = \begin{cases} R_0 (\frac{R_1}{R_2})^n & n \text{ even} \\ \frac{R_1 R_2}{R_0} (\frac{R_1}{R_2})^n & n \text{ odd} \end{cases}$$

$$\begin{aligned} g) \quad q_T(R_1) &= q \sum_{n=1}^{\infty} \left[ (\frac{R_1}{R_2})^n - \frac{R_1}{R_0} (\frac{R_1}{R_2})^{n-1} \right] \\ &= -q \frac{(R_2 - R_0)}{R_0} \sum_{n=1}^{\infty} (\frac{R_1}{R_2})^n \\ &\quad \underbrace{R_1 / (R_2 - R_1)} \\ &= \frac{-q R_1 (R_2 - R_0)}{R_0 (R_2 - R_1)} \end{aligned}$$

$$h) \quad q'_{n+1} = \frac{-R_2 q_n}{b_n} = \begin{cases} \frac{-R_2 q \left(\frac{R_1}{R_2}\right)^{n/2}}{R_o \left(\frac{R_1}{R_2}\right)^n} = \frac{-q R_2}{R_o} \left(\frac{R_2}{R_1}\right)^{n/2} & n \text{ even} \\ \frac{+ \frac{R_2 R_1}{R_o} q \left(\frac{R_1}{R_2}\right)^{(n-1)/2}}{\frac{R_1 R_2}{R_o} \left(\frac{R_1}{R_2}\right)^n} = q \left(\frac{R_2}{R_1}\right)^{(n+1)/2} & n \text{ odd} \end{cases}$$

$$b'_{n+1} = \frac{R_2^2}{b_n} = \begin{cases} \frac{R_2^2}{R_o} \left(\frac{R_2}{R_1}\right)^n & n \text{ even} \\ \frac{R_2 R_o}{R_1} \left(\frac{R_2}{R_1}\right)^n & n \text{ odd} \end{cases}$$

$$i) \quad q_T(R_2) = \sum_{n=0}^{\infty} q'_{n+1} = \frac{-q R_2}{R_o} \sum_{n=0}^{\infty} \left(\frac{R_2}{R_1}\right)^{n/2}$$

However, since  $R_2/R_1 > 1$  the series is non-convergent.

j) From Gauss's law  $\oint_S \epsilon_o \vec{E} \cdot d\vec{S} = Q_T$  enclosed. If the surface is taken inside the outer spherical conductor where  $\vec{E} = 0$  we have that the total charge enclosed is zero.

$$q_T(R_2) + q_T(R_1) + q = 0 \rightarrow q_T(R_2) = -q - q_T(R_1) = -q \left[ 1 - \frac{R_1(R_2 - R_o)}{R_o(R_2 - R_1)} \right]$$

$$= -q \frac{R_2(R_o - R_1)}{R_o(R_2 - R_1)}$$

$$k) \quad d = R_2 - R_1, \quad a = R_2 - R_o, \quad b = R_o - R_1$$

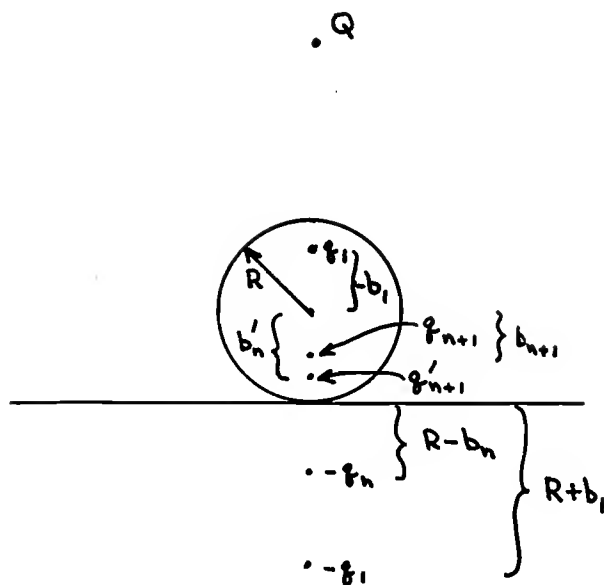
$$q_T(R_1) = \frac{-q R_1 a}{R_o d}, \quad q_T(R_2) = \frac{-q R_2 b}{R_o d}$$

$$\lim_{R_1/R_o \rightarrow 1} q_T(R_1) = \frac{-qa}{d}$$

$$R_2/R_o \rightarrow 1 \quad q_T(R_2) = \frac{-qb}{d}$$

# THE ELECTRIC FIELD

44. a)



$$q_1 = -\frac{QR}{D-R}, \quad b_1 = \frac{-R^2}{D-R}$$

$$q_{n+1} = \frac{q_n R}{2R - b_n}, \quad b_{n+1} = \frac{R^2}{2R - b_n}$$

$$q'_{n+1} = \frac{q'_n R}{2R - b'_n}, \quad b'_{n+1} = \frac{R^2}{2R - b'_n}$$

$$q'_1 = \frac{QR}{D+R}, \quad b'_1 = \frac{R^2}{D+R}$$

$-Q$

$$q_{n+1} = \frac{q_n R}{2R - b_n}, \quad b_{n+1} = \frac{R^2}{2R - b_n}$$

(Charges induced by Q)

$$(q_1 = -\frac{QR}{D-R}, \quad b_1 = \frac{-R^2}{D-R})$$

$$q'_{n+1} = \frac{q'_n R}{2R - b'_n}, \quad b'_{n+1} = \frac{R^2}{2R - b'_n}$$

[Charges induced by  $-Q$  (induced in plane

$$\text{by } Q), \quad q'_1 = \frac{QR}{R+D}, \quad b'_1 = \frac{R^2}{D+R}]$$

$$b) \quad q_{n+1} = \frac{q_n R}{2R - b_n} = \frac{q_n}{R} b_{n+1} \rightarrow q_n = \frac{q_{n-1}}{R} b_n = q_{n-1} \left(2 - \frac{q_n}{q_{n+1}}\right)$$

$$\frac{q_n}{q_{n-1}} + \frac{q_n}{q_{n+1}} - 2 = 0 \rightarrow \frac{1}{q_{n+1}} - \frac{2}{q_n} + \frac{1}{q_{n-1}} = 0$$

$$c) \quad \text{Letting } P_n = \frac{1}{q_n}$$

$$P_{n+1} - 2P_n + P_{n-1} = 0 \rightarrow P_n = A\lambda^n$$

# THE ELECTRIC FIELD

$$A\lambda^{n-1}[\lambda^2 - 2\lambda + 1] = 0 \rightarrow \lambda = +1 \quad (\text{double root})$$

$$d) \quad P_n = A_1 + A_2 n$$

First set of image charges

$$\left. \begin{aligned} P_1 &= \frac{1}{q_1} = \frac{-(D-R)}{QR} = A_1 + A_2 \\ P_2 &= \frac{1}{q_2} = \frac{2R + \frac{R^2}{D-R}}{q_1 R} = A_1 + 2A_2 \end{aligned} \right\} \rightarrow \begin{aligned} A_1 &= \frac{-R}{q_1(D-R)} = \frac{1}{Q} \\ A_2 &= \frac{D}{q_1(D-R)} = \frac{-D}{RQ} \end{aligned}$$

$$P_n = \frac{1}{Q} \left(1 - \frac{nD}{R}\right)$$

$$q_n = \frac{1}{P_n} = \frac{Q}{1 - \frac{nD}{R}}$$

Second set of image charges

$$P'_n = (A'_1 + nA'_2)$$

$$\left. \begin{aligned} P'_1 &= \frac{1}{q'_1} = \frac{D+R}{RQ} = (A'_1 + A'_2) \\ P'_2 &= \frac{1}{q'_2} = \frac{2R - \frac{R^2}{D+R}}{q'_1 R} = (A'_1 + 2A'_2) \end{aligned} \right\} \rightarrow \begin{aligned} A'_1 &= \frac{R}{(D+R)q'_1} = \frac{1}{Q} \\ A'_2 &= \frac{1}{q'_1} \frac{D}{D+R} = \frac{D}{RQ} \end{aligned}$$

$$P'_n = \frac{1}{Q} \left[1 + \frac{nD}{R}\right]$$

$$q'_n = \frac{1}{P'_n} = \frac{Q}{\left[1 + \frac{nD}{R}\right]}$$

$$\begin{aligned} b_n &= \frac{-q_n}{q_{n+1}} R + 2R \\ &= \frac{-R\left[1 - \frac{(n+1)D}{R}\right]}{\left[1 - \frac{nD}{R}\right]} + 2R \\ &= R \left[ \frac{2 - \left(1 - \frac{(n+1)D}{R}\right)}{\left(1 - \frac{nD}{R}\right)} \right] \\ &= \frac{R\left[1 - \frac{D}{R}(n-1)\right]}{\left(1 - \frac{nD}{R}\right)} \end{aligned}$$

$$b'_n = \frac{-q'_n}{q'_{n+1}} R + 2R$$

$$= \frac{R(1 + \frac{D}{R}(n-1))}{(1 + \frac{nD}{R})}$$

$$e) \quad q_T = \sum_{n=1}^{\infty} (q_n + q'_n)$$

$$= Q \sum_{n=1}^{\infty} \left( \frac{1}{1 - \frac{nD}{R}} + \frac{1}{1 + \frac{nD}{R}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{2Q}{1 - (\frac{D}{R})^2 n^2} = \sum_{n=1}^{\infty} \frac{A}{[1 - an^2]}; \quad A = 2Q, \quad a = (\frac{D}{R})^2$$

$$f) \quad \lim_{D \rightarrow \infty} E_z = \frac{-2Q}{4\pi\epsilon_0 D^2} = -E_0 \rightarrow \frac{Q}{D^2} = 2\pi\epsilon_0 E_0$$

$$g) \quad \lim_{D \rightarrow \infty} q_T = - \sum_{n=1}^{\infty} \frac{2Q}{D^2} \frac{R^2}{n^2} = -4\pi\epsilon_0 E_0 R^2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\pi^2/6}$$

$$\frac{Q}{D^2} = 4\pi\epsilon_0 E_0$$

$$q_T = -4\pi\epsilon_0 E_0 R^2 \frac{\pi^2}{6}$$

45. a) Because the image charge for a conducting sphere is less than the inducing charge it is impossible to solve this problem with only two image charges as we were able to do for parallel cylinders in (2.6). It will now require an infinite set of image charges to meet the boundary conditions.

To make the sphere an equipotential surface, place a point charge  $q_1$  at the center of the sphere. To keep the plane interface at zero potential an image charge  $-q_1$  must appear symmetrically a distance  $D$  behind the interface. However, this charge now results in an image charge  $q_2 = q_1 R/2D$  at a position  $b_2 = R^2/2D$  from the sphere center. This charge  $q_2$  must then have an image charge  $-q_2$  a distance  $D - b_2$  back from the plane interface. Then again, an image charge  $q_3 = q_2 R/(2D - b_2)$  a distance  $b_3 = R^2/(2D - b_2)$  is necessary to keep the sphere at constant potential. This process continues on for an infinite number of positive image charges confined within the sphere and negative charges within the plane conductor. Each successive image charge gets smaller in magnitude and closer to the sphere surface and closer to the plane boundary. The  $n$ th image charge is related to the previous one as

$$q_n = \frac{q_{n-1} R}{2D - b_{n-1}}, \quad b_n = \frac{R^2}{2D - b_{n-1}}$$

# THE ELECTRIC FIELD

$$q_1 = 4\pi\epsilon_0 R V_0, \quad q_2 = \frac{q_1 R}{2D} = \frac{4\pi\epsilon_0 R^2 V_0}{2D}$$

$$b) \quad \frac{q_n}{q_{n-1}} = \frac{R}{2D - b_{n-1}} = \frac{b_n}{R} \rightarrow b_n = \frac{R q_n}{q_{n-1}}$$

Letting  $n \rightarrow n + 1$

$$\frac{q_{n+1}}{q_n} = \frac{R}{2D - b_n} \rightarrow \frac{q_n}{q_{n+1}} = \frac{2D - b_n}{R} = \frac{2D - R q_n / q_{n-1}}{R}$$

$$\frac{q_n}{q_{n+1}} + \frac{q_n}{q_{n-1}} = \frac{2D}{R} \equiv C$$

$$c) \quad \text{Letting } P_n = \frac{1}{q_n}$$

$$\frac{1}{q_{n+1}} + \frac{1}{q_{n-1}} - \frac{C}{q_n} = 0 \rightarrow P_{n+1} - C P_n + P_{n-1} = 0$$

$$\text{Let } P_n = A \lambda^n$$

$$A \lambda^{n-1} [\lambda^2 - C \lambda + 1] = 0 \rightarrow \lambda^2 - C \lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{C}{2} \pm \sqrt{\left(\frac{C}{2}\right)^2 - 1} \rightarrow \lambda_2 = \frac{1}{\lambda_1}$$

$$P_n = A_1 \lambda^n + A_2 \lambda^{-n}$$

$$P_1 = \frac{1}{q_1} = A_1 \lambda + \frac{A_2}{\lambda} \quad \left. \vphantom{P_1} \right\} \rightarrow A_1 = -A_2 = \frac{\lambda}{q_1 (\lambda^2 - 1)}$$

$$P_2 = \frac{1}{q_2} = \frac{C}{q_1} = A_1 \lambda^2 + \frac{A_2}{\lambda^2}$$

$$q_n = \frac{1}{P_n} = \frac{\lambda^n}{A_1 \lambda^{2n} + A_2} = \frac{q_1 (\lambda^2 - 1)}{\lambda} \frac{\lambda^n}{(\lambda^{2n} - 1)}$$

$$d) \quad b_n = 2d - \frac{q_n R}{q_{n+1}} = 2d - \frac{R (\lambda^{2(n+1)} - 1)}{\lambda (\lambda^{2n} - 1)}$$

$$\lim_{n \rightarrow \infty} b_n = d \{1 - [1 - (\frac{R}{d})^2]^{1/2}\}$$



# THE ELECTRIC FIELD

$$\begin{aligned}
 \text{e) } C &= \frac{q_1 + q_2 + q_3 + \dots}{V_0} = \frac{\sum_{n=1}^{\infty} q_n}{V_0} \\
 &= \frac{4\pi\epsilon R}{q_1} \sum_{n=1}^{\infty} q_n \\
 &= \frac{4\pi\epsilon R(\lambda^2 - 1)}{\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{\lambda^{2n} - 1} \\
 &= 4\pi\epsilon R(\lambda^2 - 1) \left\{ \frac{1}{\lambda^2 - 1} + \frac{\lambda}{\lambda^4 - 1} + \frac{\lambda^2}{\lambda^6 - 1} + \frac{\lambda^3}{\lambda^8 - 1} + \dots \right\}
 \end{aligned}$$

→  $C_0 = 4\pi\epsilon R$  (Capacitance of an isolated sphere)

$$\lambda = \frac{C}{2} - \left[ \left( \frac{C}{2} \right)^2 - 1 \right]^{1/2}, \quad C = \frac{2D}{R}$$

$$\text{f) } q'_n = \frac{-R_2 q_n}{D \mp b'_n}, \quad b'_n = \frac{R_2^2}{D \mp b_n} \quad (\text{Upper signs for adjacent spheres, lower signs for smaller sphere } R_1 \text{ inside larger one})$$

$$q_{n+1} = \mp \frac{q'_n R_1}{D - b'_n} = \mp \frac{R_1 R_2 q_n}{D(D \mp b_n) - R_2^2}$$

$$b_{n+1} = \mp \frac{R_1^2}{D - b'_n} = \mp \frac{R_1^2 (D \mp b_n)}{D(D \mp b_n) - R_2^2} = \frac{R_1}{R_2} \frac{q_{n+1}}{q_n} (D \mp b_n)$$

g) Eliminating  $(D - b_n)$  in the unprimed variables

$$\frac{q_n}{q_{n+1}} = \mp \frac{D}{R_1^2} \frac{q_n}{q_{n+1}} b_{n+1} \mp \frac{R_2}{R_1}$$

Dropping the index by 1

$$\frac{q_{n-1}}{q_n} = \mp \frac{D}{R_1^2} \frac{q_{n-1}}{q_n} b_n \mp \frac{R_2}{R_1}$$

Since

$$b_n = -\frac{1}{D} \left[ \mp \frac{R_1 R_2 q_n}{q_{n+1}} + R_2^2 - D^2 \right]$$

we have

$$\frac{1}{q_{n+1}} + \frac{1}{q_{n-1}} + \frac{C}{q_n} = 0; \quad C = \frac{D^2 - R_1^2 - R_2^2}{R_1 R_2}$$

Letting  $P_n = \frac{1}{q_n}$

$$P_{n+1} + C P_n + P_{n-1} = 0 \rightarrow P_n = \frac{1}{q_n} = A_1 \lambda^n + A_2 \lambda^{-n}$$

h)  $\lambda = \pm \frac{C}{2} - \left[ \left( \frac{C}{2} \right)^2 - 1 \right]^{1/2}$

$$\frac{1}{q_1} = A_1 \lambda + \frac{A_2}{\lambda}$$

$$\frac{1}{q_2} = \pm \frac{(D^2 - R_2^2)}{R_1 R_2 q_1} = \pm \frac{1}{q_1} \left( C + \frac{R_1}{R_2} \right) = A_1 \lambda^2 + \frac{A_2}{\lambda^2}$$

$$A_1 = \mp \frac{A_2 \left( \lambda \pm \frac{R_1}{R_2} \right)}{\lambda \left( \pm 1 + \frac{R_1}{R_2} \right)} = -A_2 \left[ \frac{(\lambda R_2 \pm R_1)}{D \lambda} \right]^2 = -A_2 \xi^2$$

i)  $q_n = \frac{q_1 (1 - \xi^2)}{\lambda} \frac{\lambda^n}{1 - \xi^2 \lambda^{2(n-1)}}$

$$C = \frac{1}{V_o} \sum_{n=1}^{\infty} q_n = \frac{4\pi\epsilon_o R_1}{q_1} \sum_{n=1}^{\infty} q_n$$

$$= \frac{4\pi\epsilon_o R_1}{\lambda} (1 - \xi^2) \sum_{n=1}^{\infty} \frac{\lambda^n}{(1 - \xi^2 \lambda^{2(n-1)})}$$

$$= 4\pi\epsilon_o R_1 (1 - \xi^2) \left\{ \frac{1}{1 - \xi^2} + \frac{\lambda}{1 - \xi^2 \lambda^2} + \frac{\lambda^2}{1 - \xi^4 \lambda^4} + \dots \right\}$$

j) When  $D = 0 \rightarrow \lambda = \frac{R_1}{R_2} \rightarrow \xi = 0$

$$C = 4\pi\epsilon_o R_1 \{1 + \lambda + \lambda^2 + \lambda^3 + \dots\}$$

$$= \frac{4\pi\epsilon_o R_1}{1 - \lambda} = \frac{4\pi\epsilon_o R_1}{1 - \frac{R_1}{R_2}} = \frac{4\pi\epsilon_o R_1 R_2}{R_2 - R_1}$$

# CHAPTER 3 POLARIZATION AND CONDUCTION

## Section 3.1

$$1. \quad a) \quad V = \frac{-\lambda}{2\pi\epsilon_0} \ln \frac{r_1}{r_2}; \quad r_1 = [r^2 + (\frac{d}{2})^2 - r d \sin\phi]^{1/2}$$

$$r_2 = [r^2 + (\frac{d}{2})^2 + r d \sin\phi]^{1/2}$$

b) For  $r \gg d$

$$r_1 \approx r(1 - \frac{d}{2r} \sin\phi), \quad r_2 \approx r(1 + \frac{d}{2r} \sin\phi)$$

$$V \approx \frac{-\lambda}{2\pi\epsilon_0} [\ln(1 - \frac{d}{2r} \sin\phi) - \ln(1 + \frac{d}{2r} \sin\phi)]$$

$$\approx \frac{\lambda d \sin\phi}{2\pi\epsilon_0 r}$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r} \vec{i}_r - \frac{1}{r} \frac{\partial V}{\partial \phi} \vec{i}_\phi$$

$$= -\frac{\lambda d}{2\pi\epsilon_0 r^2} [\sin\phi \vec{i}_r - \cos\phi \vec{i}_\phi]$$

Let  $p = \lambda d$  (per unit length).

$$c) \quad \frac{dr}{r d\phi} = \frac{E_r}{E_\phi} = -\tan\phi \rightarrow \ln r = \ln \cos\phi + \text{constant}$$

$$\frac{r}{\cos\phi} = \text{constant}$$

$$2. \quad a) \quad \vec{p} = \int \vec{r} dq$$

$$p_z = \int_{-L}^0 -\lambda_0 z dz + \int_0^L \lambda_0 z dz = -\frac{\lambda_0 z^2}{2} \Big|_{-L}^0 + \frac{\lambda_0 z^2}{2} \Big|_0^L = \lambda_0 L^2$$

$$b) \quad p_z = \int_{-L}^0 -\lambda_0 (1 + \frac{z}{L}) z dz + \int_0^L \lambda_0 (1 - \frac{z}{L}) z dz$$

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$$= -\lambda_o \left( \frac{z^2}{2} + \frac{z^3}{3L} \right) \Big|_{-L}^0 + \lambda_o \left( \frac{z^2}{2} - \frac{z^3}{3L} \right) \Big|_0^L$$

$$= \lambda_o L^2 - \frac{2\lambda_o L^2}{3} = \frac{1}{3} \lambda_o L^2$$

$$\begin{aligned} \text{c) } p_z &= \int_0^L \lambda_o z dz = \frac{\lambda_o L^2}{2} \\ p_y &= \int_0^L -\lambda_o y dy = -\frac{\lambda_o L^2}{2} \end{aligned} \left. \vphantom{\int_0^L} \right\} \rightarrow \bar{p} = p_y \bar{i}_y + p_z \bar{i}_z = \frac{\lambda_o L^2}{2} (\bar{i}_z - \bar{i}_y)$$

$$\begin{aligned} \text{d) } p_z &= \int_{d/2}^{(d/2)+L} \lambda_o z dz - \int_{-L-(d/2)}^{-d/2} \lambda_o z dz \\ &= \frac{\lambda_o z^2}{2} \Big|_{d/2}^{(d/2)+L} - \frac{\lambda_o z^2}{2} \Big|_{-(L+\frac{d}{2})}^{-d/2} \end{aligned}$$

$$= \lambda_o \left[ \left( \frac{d}{2} + L \right)^2 - \left( \frac{d}{2} \right)^2 \right]$$

$$= \lambda_o L(L + d)$$

$$\text{e) } \bar{p} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} R \bar{i}_r \frac{Q}{2\pi} \sin\theta d\theta d\phi - \int_{\phi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} R \bar{i}_r \frac{Q}{2\pi} \sin\theta d\theta d\phi$$

$$\bar{i}_r = \sin\theta \cos\phi \bar{i}_x + \sin\theta \sin\phi \bar{i}_y + \cos\theta \bar{i}_z$$

$$p_x = p_y = 0$$

$$p_z = QR \left[ \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta d\theta - \int_{\theta=\pi/2}^{\pi} \sin\theta \cos\theta d\theta \right]$$

$$= QR \left[ \frac{\sin^2\theta}{2} \Big|_{\theta=0}^{\pi/2} - \frac{\sin^2\theta}{2} \Big|_{\theta=\pi/2}^{\pi} \right]$$

$$= QR$$

$$f) \quad dp_z = \frac{Q}{\frac{2}{3} \pi R^3} 2\pi r^3 = \frac{3Qr^3 dr}{R^3}$$

$$p_z = \frac{3Q}{R^3} \int_0^R r^3 dr$$

$$= \frac{3}{4} QR$$

$$3. \quad v = \frac{1}{4\pi\epsilon_o} \left[ -\frac{Q}{r_1} + \frac{2Q}{r} - \frac{Q}{r_2} \right]$$

$$\frac{1}{r_1} = \frac{1}{[r^2 + d^2 - 2rd\cos\theta]^{1/2}} = \frac{1}{r[1 + \frac{d}{r} [(\frac{d}{r}) - 2\cos\theta]]^{1/2}}$$

$$\approx \frac{1}{r} \left\{ 1 - \frac{1}{2} \left(\frac{d}{r}\right) \left[\left(\frac{d}{r}\right) - 2\cos\theta\right] - \frac{3}{4} \left(\frac{d}{r}\right)^2 \left[\left(\frac{d}{r}\right) - 2\cos\theta\right]^2 \right\}$$

$$\approx \frac{1}{r} \left\{ 1 + \left(\frac{d}{r}\right) \cos\theta - \frac{1}{2} \left(\frac{d}{r}\right)^2 [1 - 3\cos^2\theta] \right\}$$

Similarly

$$\frac{1}{r_2} = \frac{1}{[r^2 + d^2 + 2rd\cos\theta]^{1/2}} = \frac{1}{r[1 + \frac{d}{r} [(\frac{d}{r}) + 2\cos\theta]]^{1/2}}$$

$$\approx \frac{1}{r} \left\{ 1 - \frac{d}{r} \cos\theta - \frac{1}{2} \left(\frac{d}{r}\right)^2 [1 - 3\cos^2\theta] \right\}$$

Then

$$v \approx \frac{Q}{4\pi\epsilon_o r} \left[ \left(\frac{d}{r}\right)^2 (1 - 3\cos^2\theta) \right]$$

$$\approx \frac{Qd^2}{4\pi\epsilon_o r^3} (1 - 3\cos^2\theta)$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r} \vec{i}_r - \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{i}_\theta$$

$$= \frac{Qd^3}{4\pi\epsilon_o r^4} [3(1 - 3\cos^2\theta) \vec{i}_r - 6\cos\theta \sin\theta \vec{i}_\theta]$$

$$4. \quad a) \quad \rho = -\rho_o \left(1 - \frac{r}{R_o}\right) \quad r < R_o$$

$$-Q = -4\pi \int_0^{R_o} \rho_o \left(1 - \frac{r}{R_o}\right) r^2 dr$$

$$= -4\pi \rho_o \left[ \frac{R_o^3}{3} - \frac{R_o^4}{4R_o} \right]$$

$$= -\frac{\pi \rho_o R_o^3}{3} \rightarrow \rho_o = \frac{3Q}{\pi R_o^3}$$

b) The electric field due to the volume charge distribution is

$$\oint_S \epsilon_o \vec{E} \cdot d\vec{S} = \epsilon_o E_r 4\pi r^2 = -4\pi \rho_o \int_0^r \left(1 - \frac{r}{R_o}\right) r^2 dr$$

$$= -4\pi \rho_o \left[ \frac{r^3}{3} - \frac{r^4}{4R_o} \right]$$

$$E_r = -\frac{\rho_o}{\epsilon_o} \left[ \frac{r}{3} - \frac{r^2}{4R_o} \right] \quad r < R_o$$

$$\bar{F} = Q \left[ E_{LOC} - \frac{\rho_o}{\epsilon_o} \left( \frac{d}{3} - \frac{d^2}{4R_o} \right) \right] = 0$$

$$d^2 - \frac{4R_o d}{3} + \frac{4R_o \epsilon_o E_{LOC}}{\rho_o} = 0$$

$$d = \frac{2}{3} R_o - \sqrt{\frac{4}{9} R_o^2 - \frac{4R_o \epsilon_o E_{LOC}}{\rho_o}}$$

$$= \frac{2}{3} R_o \left[ 1 - \sqrt{1 - \frac{9\epsilon_o E_{LOC}}{\rho_o R_o}} \right]$$

$$c) \quad \frac{9\epsilon_o E_{LOC}}{\rho_o R_o} \ll 1$$

$$d \approx \frac{2}{3} R_o \left[ \frac{9}{2} \frac{\epsilon_o E_{LOC}}{\rho_o R_o} \right]$$

$$= \frac{3\epsilon_o E_{LOC}}{\rho_o}$$

$$\bar{p} = \alpha \bar{E}_{LOC} = Qd \rightarrow \alpha = \frac{3\epsilon_o Q}{\rho_o} = \epsilon_o \pi R_o^3$$

$$\begin{aligned} 5. \quad a) \quad \bar{E}_T &= E_o \bar{i}_z + \frac{2\bar{p}}{4\pi\epsilon_o a^3} \\ &= E_o \bar{i}_z + \frac{2\alpha\bar{E}_T}{4\pi\epsilon_o a^3} \rightarrow \bar{E}_T = \frac{E_o \bar{i}_z}{[1 - \frac{\alpha}{2\pi\epsilon_o a^3}]} \end{aligned}$$

$$\begin{aligned} b) \quad \bar{E}_T &= E_o \bar{i}_z + \frac{2\alpha\bar{E}_T}{4\pi\epsilon_o a^3} \sum_{n=1}^{\infty} \frac{2}{n^3} \\ &\approx E_o \bar{i}_z + \frac{1.2\alpha\bar{E}_T}{\pi\epsilon_o a^3} \rightarrow \bar{E}_T = \frac{E_o \bar{i}_z}{[1 - \frac{1.2\alpha}{\pi\epsilon_o a^3}]} \end{aligned}$$

$$c) \quad \bar{p} = \alpha \bar{E}_T = \frac{\alpha E_o \bar{i}_z}{[1 - \frac{1.2\alpha}{\pi\epsilon_o a^3}]}$$

$$\bar{P} = N\bar{p} = \frac{\bar{p}}{a^3} = \frac{\alpha E_o \bar{i}_z}{a^3 [1 - \frac{1.2\alpha}{\pi\epsilon_o a^3}]} = \chi_e \epsilon_o \bar{E}_o$$

$$\chi_e = \frac{\alpha}{\epsilon_o a^3 [1 - \frac{1.2\alpha}{\pi\epsilon_o a^3}]} \rightarrow \epsilon = \epsilon_o (1 + \chi_e)$$

6. a) The electric field within the sphere due to the volume charge distribution is

$$\bar{E}_- = \frac{-Qr}{4\pi\epsilon_o R_o^3} \bar{i}_r$$

The electric field due to the point charge with reference to a spherical coordinate system with origin at the center of the sphere is

$$\bar{E}_+ = \frac{Q \bar{i}_{QP}}{4\pi\epsilon_o r_{QP}^2} = \frac{Q(r\bar{i}_r - d\bar{i}_z)}{4\pi\epsilon_o [r^2 + d^2 - 2rd\cos\theta]^{3/2}}$$

The total dipole electric field is then

$$\bar{E}_{dip} = \bar{E}_- + \bar{E}_+ = \frac{-Qr\bar{i}_r}{4\pi\epsilon_o R_o^3} + \frac{Q(r\bar{i}_r - d\bar{i}_z)}{4\pi\epsilon_o [r^2 + d^2 - 2rd\cos\theta]^{3/2}}$$

$$b) \bar{i}_r = \sin\theta\cos\phi\bar{i}_x + \sin\theta\sin\phi\bar{i}_y + \cos\theta\bar{i}_z$$

$$\langle E_x \rangle_{\text{dip}} = \frac{1}{\frac{4}{3}\pi R_o^3} \int_0^R \int_0^\pi \int_0^{2\pi} E_{x\text{dip}} r^2 \sin\theta dr d\theta d\phi$$

Since  $E_{x\text{dip}}$  is proportional to  $\cos\phi$  which integrates to zero over  $\phi$

$$\int_0^{2\pi} \cos\phi = 0$$

the average x component of field is zero. Similarly  $E_y$  is proportional to  $\sin\phi$  which also integrates to zero over  $\phi$ .

$$c) E_{z\text{dip}} = -\frac{Qr\cos\theta}{4\pi\epsilon_o R^3} + \frac{Q(r\cos\theta - d)}{4\pi\epsilon_o [r^2 + d^2 - 2rd\cos\theta]^{3/2}}$$

so that after integrating over  $\phi$

$$\langle E_z \rangle_{\text{dip}} = \frac{2\pi}{\frac{4}{3}\pi R_o^3} \int_0^R \int_0^\pi \frac{Q}{4\pi\epsilon_o} \left[ -\frac{r\cos\theta}{R_o^3} + \frac{(r\cos\theta - d)}{[r^2 + d^2 - 2rd\cos\theta]^{3/2}} \right] r^2 \sin\theta dr d\theta$$

The first term in the integrand integrates to zero over  $\theta$ . The second term is integrated by introducing the change of variable

$$u = r^2 + d^2 - 2rd\cos\theta, du = 2rdsin\theta d\theta$$

so that

$$\langle E_z \rangle_{\text{dip}} = \frac{3Q}{8\pi\epsilon_o R_o^3} \int_0^R \int_{u=(r-d)^2}^{(r+d)^2} \frac{r(r\cos\theta - d)}{2du^{3/2}} dr du$$

Now

$$r\cos\theta = -\frac{(u - r^2 - d^2)}{2d}$$

so that the integral reduces to

$$\begin{aligned} \langle E_z \rangle_{\text{dip}} &= \frac{3Q}{32\pi\epsilon_o R_o^3 d^2} \int_0^R \int_{u=(r-d)^2}^{(r+d)^2} \left[ \frac{-r}{u^{1/2}} + \frac{r(r^2 - d^2)}{u^{3/2}} \right] dr du \\ &= \frac{3Q}{32\pi\epsilon_o R_o^3 d^2} \int_0^R \left[ -2ru^{1/2} - \frac{2r(r^2 - d^2)}{u^{1/2}} \right] \bigg|_{u=(r-d)^2}^{(r+d)^2} dr \end{aligned}$$



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$$= \frac{-3Q}{16\pi\epsilon_o R_o^3 d^2} \int_{r=0}^R r[r+d - |r-d| + r-d - \frac{(r+d)(r-d)}{|r-d|}] dr$$

When  $r$  is greater than  $d$  the integrand is zero so that the integral reduces to

$$\langle E_z \rangle_{\text{dip}} = \frac{-3Q}{4\pi\epsilon_o R_o^3 d^2} \int_{r=0}^d r^2 dr$$

$$= \frac{-Qd}{4\pi\epsilon_o R_o^3} = \frac{-p_z}{4\pi\epsilon_o R_o^3}$$

$$d) \langle \bar{E} \rangle_{\text{dip}} = \frac{-\bar{p}}{4\pi\epsilon_o R_o^3} = \frac{-\bar{P}}{3\epsilon_o}$$

$$\bar{P} = N\alpha \bar{E}_{\text{LOC}} = N\alpha [\bar{E} - \langle \bar{E} \rangle_{\text{dip}}] = N\alpha [\bar{E} + \frac{\bar{P}}{3\epsilon_o}]$$

$$\bar{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_o}} \bar{E}$$

$$7. a) \bar{d} = \frac{4\pi\epsilon_o R_o^3}{Q} \bar{E}_o$$

$$b) m \frac{d^2 \bar{d}}{dt^2} = Q [\bar{E}_o - \frac{Q\bar{d}}{4\pi\epsilon_o R_o^3}]$$

$$\text{If } \bar{E}_o = 0$$

$$\frac{d^2 \bar{d}}{dt^2} + \frac{Q^2}{4\pi\epsilon_o R_o^3 m} \bar{d} = 0$$

$$\bar{d} = \bar{d}_o \cos \omega_o t; \quad \bar{d}_o = \frac{4\pi\epsilon_o R_o^3}{Q} \bar{E}_o, \quad \omega_o = \sqrt{\frac{Q^2}{4\pi\epsilon_o R_o^3 m}}$$

$$c) Q = e = 1.6 \times 10^{-19} \text{ Coulombs}, R_o = 10^{-10} \text{ meter}, m = 9.1 \times 10^{-31} \text{ kg.}$$

$$\omega_o = 1.6 \times 10^{16} \text{ rad/sec} \rightarrow f_o = \frac{\omega_o}{2\pi} = 2.5 \times 10^{15} \text{ Hz}$$

$$d) m \frac{d^2 \bar{d}}{dt^2} = -\beta \frac{d\bar{d}}{dt} - \frac{Q^2}{4\pi\epsilon_o R_o^3} \bar{d} + Q \text{Re}(\bar{E}_o e^{j\omega t})$$

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$$\frac{d^2 \bar{d}}{dt^2} + \gamma \frac{d\bar{d}}{dt} + \omega_o^2 \bar{d} = \frac{Q}{m} \operatorname{Re}(\bar{E}_o e^{j\omega t}); \quad \gamma = \frac{\beta}{m}$$

e)  $\bar{d} = \operatorname{Re} \hat{d} e^{j\omega t}$

$$\hat{d} = \frac{QE_o/m}{[-\omega^2 + \gamma j\omega + \omega_o^2]}$$

$$\hat{p} = Q\hat{d} = \frac{Q^2 E_o/m}{[-\omega^2 + \gamma j\omega + \omega_o^2]} = \hat{\alpha} E_o \rightarrow \hat{\alpha} = \frac{Q^2/m}{[-\omega^2 + \gamma j\omega + \omega_o^2]}$$

f)  $\hat{\epsilon} = \epsilon_o \left[ 1 + \frac{N\hat{\alpha}/\epsilon_o}{1 - \frac{N\hat{\alpha}}{3\epsilon_o}} \right]$

$$= \epsilon_o \left[ 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2 + \gamma j\omega - \frac{\omega_p^2}{3}} \right]; \quad \omega_p^2 = \frac{Q^2 N}{m\epsilon_o}$$

$$= \epsilon_o \left\{ 1 + \frac{\omega_p^2 [(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3}) - \gamma j\omega]}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]} \right\}$$

$$\epsilon_r = \epsilon_o \left\{ 1 + \frac{\omega_p^2 (\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]} \right\}$$

$$\epsilon_i = \frac{-\epsilon_o \omega_p^2 \gamma \omega}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]}$$

g) If the lossy dielectric is placed between parallel plate electrodes

$$\begin{aligned} \frac{\hat{i}}{A} &= \frac{j\omega \hat{\epsilon} \hat{V}}{s} \rightarrow Y = \frac{\hat{i}}{\hat{V}} = \frac{j\omega \hat{\epsilon} A}{s} \\ &= \frac{j\omega \epsilon_o A}{s} \left\{ 1 + \frac{\omega_p^2 [(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3}) - \gamma j\omega]}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]} \right\} \end{aligned}$$

$$= \frac{j\omega\epsilon_o A}{s} \left\{ 1 + \frac{\frac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})^2} \left[ (\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 - \gamma j\omega \right]}{\left[ 1 - \frac{\omega^2}{(\omega_o^2 - \frac{\omega_p^2}{3})} \right]^2 + \frac{\gamma^2 \omega^2}{(\omega_o^2 - \frac{\omega_p^2}{3})^2}} \right\}$$

The admittance of the equivalent circuit shown is

$$Y = j\omega C_1 \left\{ 1 + \frac{\frac{C_2}{C_1} [1 - LC_2 \omega^2 - RC_2 j\omega]}{[(1 - LC_2 \omega^2)^2 + (RC_2 \omega)^2]} \right\}$$

so that

$$C_1 = \frac{\epsilon_o A}{s}, \quad LC_2 = \frac{1}{(\omega_o^2 - \frac{\omega_p^2}{3})}, \quad RC_2 = \frac{\gamma}{(\omega_o^2 - \frac{\omega_p^2}{3})}, \quad \frac{C_2}{C_1} = \frac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})}$$

$$\rightarrow C_2 = \frac{\epsilon_o A}{s} \frac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})}, \quad L = \frac{1}{\omega_p^2 \frac{\epsilon_o A}{s}}, \quad R = \frac{\gamma}{\omega_p^2 \frac{\epsilon_o A}{s}}$$

h) When  $m \rightarrow 0$ ,  $\omega_p^2$ ,  $\omega_o^2$ , and  $\gamma \rightarrow \infty$  so that

$$\epsilon_r = \epsilon_o \left[ 1 + \frac{\frac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})} \left( \omega_o^2 - \frac{\omega_p^2}{3} \right)}{(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2} \right]$$

$\lim_{m \rightarrow 0}$

$$\epsilon_i = \frac{-\epsilon_o \omega_p^2 \gamma \omega}{[(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]}$$

From  $\epsilon_r$  we have

$$[(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2] = \frac{\omega_p^2 (\omega_o^2 - \frac{\omega_p^2}{3}) \epsilon_o}{\epsilon_r - \epsilon_o}$$

so that

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$$\epsilon_i = \frac{-\gamma\omega(\epsilon_r - \epsilon_o)}{(\omega_o^2 - \frac{\omega_p^2}{3})}$$

Squaring both sides and eliminating  $\gamma^2\omega^2$  yields

$$\epsilon_i^2 + (\epsilon_r - \epsilon_o)^2 - \frac{(\epsilon_r - \epsilon_o)\epsilon_o\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})} = 0$$

By completing the square this relation becomes

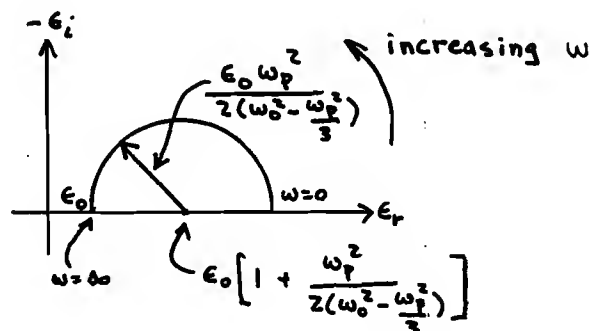
$$\epsilon_i^2 + \left[ \epsilon_r - \epsilon_o \left( 1 + \frac{\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})} \right) \right]^2 = \left[ \frac{\epsilon_o\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})} \right]^2$$

which is the equation of a circle with radius

$$r = \frac{\epsilon_o\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})}$$

and center at

$$\epsilon_i = 0, \quad \epsilon_r = \epsilon_o \left[ 1 + \frac{\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})} \right]$$



When  $m = 0$ , the equivalent circuit of (g) has no inductance.

- i) To find  $\epsilon_{i_{\max}}$

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$$\frac{\partial \epsilon_i}{\partial \omega} = \frac{-\epsilon_o \omega^2 \gamma}{2} \frac{[-2\gamma^2 \omega^2 + (\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]}{[(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2 \omega^2]^2} = 0$$

which is zero at frequency

$$\omega_{\max}^2 = \frac{(\omega_o^2 - \frac{\omega_p^2}{3})^2}{\gamma^2}$$

so that

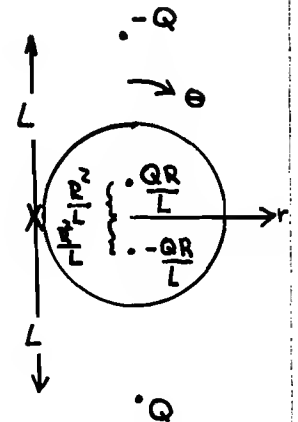
$$(\epsilon_i)_{\max} = \frac{\epsilon_o \omega_p^2 \gamma \omega_{\max}}{2(\gamma^2 \omega_{\max}^2)} = \frac{\epsilon_o \omega_p^2}{2\gamma \omega_{\max}} = \frac{\epsilon_o \omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})}$$

8. a) Each charge  $Q$  induces image charges in the sphere  $-\frac{QR}{L}$ . Along  $z$  axis

$$E_z = \frac{Q}{4\pi\epsilon_o} \left[ \frac{(z+L)}{|z+L|^3} - \frac{(z-L)}{|z-L|^3} - \frac{(z+\frac{R^2}{L})}{|z+\frac{R^2}{L}|^3} + \frac{(z-\frac{R^2}{L})}{|z-\frac{R^2}{L}|^3} \right]$$

In  $z=0$  plane

$$E_z = \frac{Q}{4\pi\epsilon_o} \left[ \frac{2L}{[L^2 + r^2]^{3/2}} - \frac{2R^3}{L^2[(\frac{R^2}{L})^2 + r^2]^{3/2}} \right]$$



b)  $\lim_{L \gg r} E_z \approx \frac{Q}{2\pi\epsilon_o} \left[ \frac{1}{L^2} \right], \quad \frac{Q}{L^2} = 2\pi\epsilon_o E_o$

- c) In this limit, the two image charges form a point dipole with moment

$$p_z = \frac{QR}{L} \frac{2R^2}{L} = \frac{2QR^3}{L^2} = 4\pi\epsilon_o R^3 E_o$$

The electric field is then the superposition of the imposed field  $E_o \bar{i}_z$  with the induced dipole field

$$\begin{aligned} \bar{E}_T &= E_o \bar{i}_z + \frac{p_z}{4\pi\epsilon_o r^3} [2\cos\theta \bar{i}_r + \sin\theta \bar{i}_\theta] \\ &= E_o [\bar{i}_r \cos\theta - \bar{i}_\theta \sin\theta] + \frac{E_o R^3}{r^3} [2\cos\theta \bar{i}_r + \sin\theta \bar{i}_\theta] \\ &= E_o \left\{ \left(1 + \frac{2R^3}{r^3}\right) \cos\theta \bar{i}_r - \left(1 - \frac{R^3}{r^3}\right) \sin\theta \bar{i}_\theta \right\} \end{aligned}$$

Note that at  $r = R$ ,  $E_\theta(r=R) = 0$ . The electric field must be normally incident onto a perfect conductor.

9. a)  $\bar{f} = q[\bar{E}(\bar{r} + \bar{d}) - \bar{E}(\bar{r})]$

$$\bar{E}(\bar{r} + \bar{d}) \approx \bar{E}(\bar{r}) + \frac{\partial \bar{E}}{\partial x} \left| \frac{d_x}{r} + \frac{\partial \bar{E}}{\partial y} \left| \frac{d_y}{r} + \frac{\partial \bar{E}}{\partial z} \left| \frac{d_z}{r} \right. \right. \right.$$

$$\approx \bar{E}(\bar{r}) + (\bar{d} \cdot \nabla) \bar{E}$$

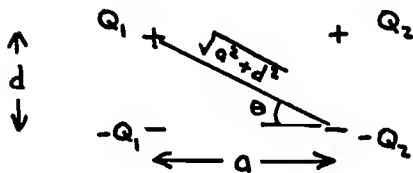
$$\bar{f} = q(\bar{d} \cdot \nabla) \bar{E} = (\bar{p} \cdot \nabla) \bar{E}$$

b) Colinear

$$E_{z2} = \frac{-p_2}{2\pi\epsilon_0 z^3}$$

$$\bar{f}_1 = p_1 \frac{\partial}{\partial z} E_{z2} \bar{i}_z = \frac{3p_1 p_2}{2\pi\epsilon_0 z^4} \left| \bar{i}_z = \frac{3p_1 p_2}{2\pi\epsilon_0 a^4} \bar{i}_z \right|_{z=a}$$

Adjacent



$$f_{1r} = \frac{2Q_1 Q_2}{4\pi\epsilon_0} \left[ \frac{-1}{a^2} + \frac{\cos\theta}{[a^2 + d^2]} \right]; \cos\theta = \frac{a}{[a^2 + d^2]^{1/2}}$$

$$= \frac{Q_1 Q_2}{2\pi\epsilon_0} \left[ -\frac{1}{a^2} + \frac{a}{[a^2 + d^2]^{3/2}} \right]$$

$$= \frac{Q_1 Q_2}{2\pi\epsilon_0} \left[ -\frac{1}{a^2} + \frac{1}{a^2 [1 + \frac{d^2}{a^2}]^{3/2}} \right]$$

$$\approx \frac{Q_1 Q_2 (-\frac{3}{2} d^2)}{2\pi\epsilon_0 a^4} = \frac{-3p_1 p_2}{4\pi\epsilon_0 a^4}$$

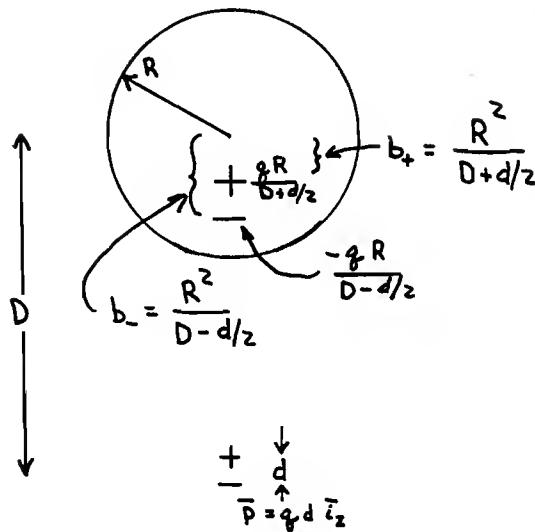
c) Colinear

$$\begin{aligned}\bar{f}_1 &= \frac{3p_1^2}{2\pi\epsilon_0 a^4} \bar{i}_z \sum_{n=1}^{\infty} \frac{1}{n^4} \\ &= \frac{\pi^3 p_1^2}{60\epsilon_0 a^4} \bar{i}_z\end{aligned}$$

Adjacent

$$\begin{aligned}\bar{f}_1 &= \frac{-3p^2}{4\pi\epsilon_0 a^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \\ &= \frac{-p^2 \pi^3}{120\epsilon_0 a^4} \bar{i}_r\end{aligned}$$

10. a)


 For  $d \ll D$ 

$$b_- - b_+ \approx \frac{R^2}{D} \left[ 1 + \frac{d}{2D} - 1 + \frac{d}{2D} \right] \approx \frac{R^2 d}{D^2}$$

while each induced charge is approximately  $\pm \frac{qR}{D}$ . The resulting induced dipole moment is then

$$\bar{p}_{\text{ind}} = \frac{qR}{D} \frac{R^2 d}{D^2} \bar{i}_z = qd \frac{R^3}{D^3} \bar{i}_z = \bar{p} \frac{R^3}{D^3}$$

b) The electric field along the  $z$  axis is just due to the superposition of fields due to both dipoles

$$E_z = \frac{p}{2\pi\epsilon_0 |z + D|^3} + \frac{p R^3/D^3}{2\pi\epsilon_0 |z + \frac{R^2}{D}|^3}$$

$$c) \quad \bar{f}_{\text{sphere}} = (\bar{p}_{\text{ind}} \cdot \nabla) \bar{E}$$

$$= \frac{pR^3}{D^3} \frac{\partial}{\partial z} \left( \frac{p}{2\pi\epsilon_0 |z + D|^3} \right) \Big|_{z = -\frac{R^2}{D}}$$

$$= \frac{-3p^2 R^3}{D^3 2\pi\epsilon_0 (z + D)^4} \Big|_{z = -\frac{R^2}{D}}$$

$$= \frac{-3p^2 R^3 D}{2\pi\epsilon_0 (D^2 - R^2)^4}$$

### Section 3.2

$$11. \quad \nabla^2 V - \frac{V^2}{\ell_d^2} = 0; \quad \ell_d^2 = \frac{\epsilon kT}{2\rho_0 q}$$

$$a) \quad \frac{d^2 V}{dx^2} - \frac{V^2}{\ell_d^2} = 0 \rightarrow V = \begin{cases} V_0 e^{-x/\ell_d} & x > 0 \\ V_0 e^{+x/\ell_d} & x < 0 \end{cases}$$

$$E_x = -\frac{dV}{dx} = \begin{cases} \frac{V_0}{\ell_d} e^{-x/\ell_d} & x > 0 \\ -\frac{V_0}{\ell_d} e^{+x/\ell_d} & x < 0 \end{cases}$$

$$\sigma_f = \epsilon [E_x(x=0_+) - E_x(x=0_-)] = \frac{2\epsilon V_0}{\ell_d} \rightarrow V_0 = \frac{\sigma_f \ell_d}{2\epsilon}$$

$$\rho_f = \frac{\epsilon dE_x}{dx} = \frac{-\sigma_f}{2\ell_d} e^{-|x|/\ell_d}$$

$$b) \quad \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) - \frac{V}{\ell_d^2} = 0 \rightarrow V = V_0 K_0(r/\ell_d)$$

$$E_r = -\frac{dV}{dr} = \frac{V_0}{\ell_d} K_1(r/\ell_d)$$



$$\lim_{r \rightarrow 0} E_r = \frac{V_o}{r} = \frac{\lambda}{2\pi\epsilon r} \rightarrow V_o = \frac{\lambda}{2\pi\epsilon}$$

$$V = \frac{\lambda}{2\pi\epsilon} K_o(r/\ell_d), \quad E_r = \frac{\lambda}{2\pi\epsilon\ell_d} K_1(r/\ell_d)$$

$$\rho_f = \frac{\epsilon}{r} \frac{\partial}{\partial r} (rE_r) = -\frac{\epsilon V}{\ell_d^2} = \frac{-\lambda}{2\pi\ell_d^2} K_o(r/\ell_d)$$

$$c) \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial V}{\partial r}) - \frac{V}{\ell_d^2} = 0 \rightarrow V = \frac{A_1}{r} e^{-r/\ell_d}$$

$$E_r = -\frac{dV}{dr} = \frac{A_1 e^{-r/\ell_d}}{r} \left( \frac{1}{r} + \frac{1}{\ell_d} \right)$$

$$E_r(r=R) = \frac{Q}{4\pi\epsilon R^2} = \frac{A_1 e^{-R/\ell_d}}{R} \left( \frac{1}{R} + \frac{1}{\ell_d} \right) \rightarrow A_1 = \frac{Q e^{R/\ell_d}}{4\pi\epsilon R \left( \frac{1}{R} + \frac{1}{\ell_d} \right)}$$

$$V = \frac{Q e^{-(r-R)/\ell_d}}{4\pi\epsilon R r \left( \frac{1}{R} + \frac{1}{\ell_d} \right)}, \quad E_r = \frac{Q e^{-(r-R)/\ell_d} \left( \frac{1}{r} + \frac{1}{\ell_d} \right)}{4\pi\epsilon R r \left( \frac{1}{R} + \frac{1}{\ell_d} \right)}$$

$$\rho_f = \frac{\epsilon}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = \frac{-Q e^{-(r-R)/\ell_d}}{4\pi\ell_d r (R + \ell_d)}$$

$$12. a) \quad \frac{d^2 V}{dx^2} - \frac{V}{\ell_d^2} = 0 \rightarrow V = V_1 e^{x/\ell_d} + V_2 e^{-x/\ell_d}$$

$$V(x = -\ell) = -\frac{V_o}{2} = V_1 e^{-\ell/\ell_d} + V_2 e^{\ell/\ell_d}$$

$$V(x = \ell) = \frac{V_o}{2} = V_1 e^{\ell/\ell_d} + V_2 e^{-\ell/\ell_d}$$

$$V_1 = -V_2 = \frac{V_o}{4 \sinh \ell/\ell_d}$$

$$V(x) = \frac{V_o}{4 \sinh \ell/\ell_d} [e^{x/\ell_d} - e^{-x/\ell_d}]$$

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$$= \frac{V_o}{2} \frac{\sinh x/\ell_d}{\sinh \ell/\ell_d}$$

$$E_x = -\frac{dV}{dx} = \frac{-V_o}{2\ell_d} \frac{\cosh x/\ell_d}{\sinh \ell/\ell_d}$$

$$\rho_f = \epsilon \frac{dE_x}{dx} = \frac{-\epsilon V_o}{2\ell_d^2} \frac{\sinh x/\ell_d}{\sinh \ell/\ell_d}$$

Positive charge near negative electrode and negative charge near positive electrode.

$$b) \quad \sigma_f(x=-\ell) = \epsilon E_x(x=-\ell) = -\frac{\epsilon V_o}{2\ell_d} \coth \ell/\ell_d$$

$$\sigma_f(x=+\ell) = -\epsilon E_x(x=+\ell) = \frac{\epsilon V_o}{2\ell_d} \coth \ell/\ell_d$$

$$\sigma_T = \int_{-\ell}^{+\ell} \rho_f dx = \frac{-\epsilon V_o}{2\ell_d^2 \sinh \ell/\ell_d} \int_{-\ell}^{+\ell} \sinh x/\ell_d dx = 0$$

$$13. \quad m \frac{dv}{dt} = -mvv + qE_o$$

$$a) \quad \frac{dv}{dt} + vv = \frac{q}{m} E_o$$

$$v = A_1 e^{-vt} + \frac{q}{mv} E_o$$

$$v(t=0) = 0 \rightarrow A_1 = -\frac{qE_o}{mv} \rightarrow v = \frac{qE_o}{mv} [1 - e^{-vt}]$$

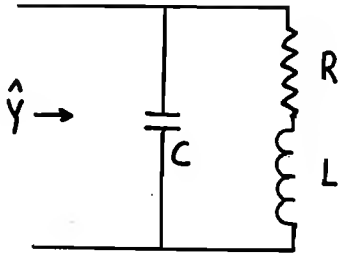
$$b) \quad v = \frac{qE_o}{mv} e^{-vt}$$

$$c) \quad v(t) = \text{Re} \hat{V} e^{j\omega t} \rightarrow (j\omega + v) \hat{V} = \frac{q \hat{E}}{m}; \quad \hat{E} = \frac{V_o}{s}$$

$$\hat{I} = (qn \hat{V} + j\omega \hat{E}) A = \left( \frac{q^2 n}{m(j\omega + v)} + j\omega \epsilon \right) \frac{AV_o}{s}$$

$$\hat{Y} = \frac{\hat{I}}{V_o} = \frac{\epsilon A}{s} \left( j\omega + \frac{\omega_p^2}{j\omega + v} \right); \quad \omega_p^2 = \frac{q^2 n}{m\epsilon}$$

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$$\hat{Y} = Cj\omega + \frac{1}{R + Lj\omega},$$

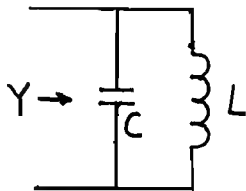
$$C = \frac{\epsilon A}{s}, \quad L = \frac{s}{\epsilon A \omega_p^2}, \quad R = \frac{vs}{\epsilon A \omega_p^2}$$

$$14. \quad J_c + \epsilon \frac{\partial E}{\partial t} = \frac{i(t)}{A} \rightarrow \frac{\partial J_c}{\partial t} + \epsilon \frac{\partial^2 E}{\partial t^2} = \frac{1}{A} \frac{di}{dt}$$

$$\frac{d^2 E}{dt^2} + \omega_{pe}^2 E = \frac{1}{\epsilon A} \frac{di}{dt}; \quad i(t) = \text{Re} \hat{I} e^{j\omega t}; \quad E = \text{Re} \hat{E} e^{j\omega t}, \quad \hat{E} = \frac{\hat{V}}{s}$$

$$\frac{\hat{V}}{s} [-\omega^2 + \omega_{pe}^2] = \frac{j\omega \hat{I}}{\epsilon A} \rightarrow \hat{I} = \frac{\epsilon A \hat{V}}{s} \frac{(\omega_{pe}^2 - \omega^2)}{j\omega}$$

$$Y = \frac{\hat{I}}{\hat{V}} = \frac{\epsilon A}{s} \left( \frac{\omega_{pe}^2 - \omega^2}{j\omega} \right) = \frac{j\omega s}{\epsilon A \omega_{pe}^2} \left( 1 - \frac{\omega^2}{\omega_{pe}^2} \right)$$



$$Y = Cj\omega + \frac{1}{Lj\omega} = \frac{1 - LC\omega^2}{Lj\omega} = \frac{\hat{I}}{\hat{V}}$$

$$L = \frac{s}{\epsilon A \omega_p^2}, \quad LC = \frac{1}{\omega_{pe}^2} \rightarrow C = \frac{1}{L\omega_{pe}^2} = \frac{\epsilon A}{s}$$

$$15. \quad a) \quad \frac{i}{A} = \sigma E$$

$$b) \quad \Delta(mv) = \int qEdt = q \int \frac{i}{A\sigma} dt = \frac{q}{A\sigma} \underbrace{\int i dt}_Q = \frac{qQ}{A\sigma} = m\omega R$$

$$Q = \frac{m\omega R A \sigma}{q}$$

$$c) \quad \frac{di}{dt} = \omega_p^2 \epsilon E A$$

$$\Delta(mv) = \frac{q}{\omega_p^2 \epsilon A} \int \frac{di}{dt} = \frac{qI}{\omega_p^2 \epsilon A} = m\omega R \rightarrow I = \frac{m\omega R \omega_p^2 \epsilon A}{q}$$

Section 3.3

16. a) For lossless dielectrics

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0 \rightarrow \epsilon_1 E_1 \cos \theta_1 = \epsilon_2 E_2 \cos \theta_2$$

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \rightarrow E_1 \sin \theta_1 = E_2 \sin \theta_2$$

$$\epsilon_1 \cot \theta_1 = \epsilon_2 \cot \theta_2 \rightarrow \frac{\epsilon_2}{\epsilon_1} = \frac{\tan \theta_2}{\tan \theta_1}$$

$$\begin{aligned} E_2 &= \frac{\epsilon_1 E_1 \cos \theta_1}{\epsilon_2 \cos \theta_2} = \frac{\epsilon_1 E_1}{\epsilon_2} \cos \theta_1 \sqrt{1 + \tan^2 \theta_2} \\ &= \frac{\epsilon_1 E_1}{\epsilon_2} \cos \theta_1 \left[ 1 + \left( \frac{\epsilon_2}{\epsilon_1} \right)^2 \tan^2 \theta_1 \right]^{1/2} \end{aligned}$$

b) For conductors

$$\vec{n} \cdot (\vec{J}_2 - \vec{J}_1) = 0 \rightarrow \sigma_1 E_1 \cos \theta_1 = \sigma_2 E_2 \cos \theta_2$$

so that we can replace the permittivities in (a) by the respective conductivities

$$\tan \theta_2 = \frac{\sigma_2}{\sigma_1} \tan \theta_1 \Rightarrow E_2 = \frac{\sigma_1 E_1 \cos \theta_1}{\sigma_2} \left[ 1 + \left( \frac{\sigma_2}{\sigma_1} \right)^2 \tan^2 \theta_1 \right]^{1/2}$$

$$\sigma_f = D_{2n} - D_{1n} = \epsilon_2 E_2 \cos \theta_2 - \epsilon_1 E_1 \cos \theta_1 = E_1 \cos \theta_1 \left[ -\epsilon_1 + \epsilon_2 \frac{\sigma_1}{\sigma_2} \right]$$

$$c) \quad \vec{n} \cdot (\vec{J}_2 - \vec{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0 \Rightarrow \vec{n} \cdot \left[ \vec{J}_2 - \vec{J}_1 + \frac{\partial}{\partial t} (\vec{D}_2 - \vec{D}_1) \right] = 0$$

$$(\sigma_2 + j\omega\epsilon_2) \hat{E}_2 \cos \theta_2 = (\sigma_1 + j\omega\epsilon_1) \hat{E}_1 \cos \theta_1$$

$$\hat{E}_2 \sin \theta_2 = \hat{E}_1 \sin \theta_1$$

$$\tan \theta_2 = \frac{(\sigma_2 + j\omega\epsilon_2) \tan \theta_1}{(\sigma_1 + j\omega\epsilon_1)}$$

$$\hat{\sigma}_f = \epsilon_2 \hat{E}_2 \cos \theta_2 - \epsilon_1 \hat{E}_1 \cos \theta_1 = \hat{E}_1 \cos \theta_1 \left( -\epsilon_1 + \frac{\epsilon_2 (\sigma_1 + j\omega\epsilon_1)}{(\sigma_2 + j\omega\epsilon_2)} \right)$$

$$17. a) \quad D_r = \frac{\lambda}{2\pi r} \quad 0 < r < \infty; \quad E_r = \begin{cases} \frac{\lambda}{2\pi\epsilon r} & 0 < r < a \\ \frac{\lambda}{2\pi\epsilon_o r} & a < r < \infty \end{cases}$$

$$P_r = (\epsilon - \epsilon_o) E_r = \begin{cases} \frac{(\epsilon - \epsilon_o) \lambda}{2\pi\epsilon r} & 0 < r < a \\ 0 & a < r < \infty \end{cases}$$

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$$\sigma_p(r=a) = -[P_r(r=a_+) - P_r(r=a_-)] = \frac{(\epsilon - \epsilon_o)\lambda}{2\pi\epsilon a}$$

$$\lambda_p(r=0) = -\frac{(\epsilon - \epsilon_o)\lambda}{\epsilon}$$

$$b) \quad D_y = \pm \frac{\sigma_f}{2} \quad \begin{matrix} y > 0 \\ y < 0 \end{matrix}; \quad E_y = \begin{cases} \pm \frac{\sigma_f}{2\epsilon} & 0 < y < \frac{d}{2} \\ -\frac{d}{2} < y < 0 \\ \pm \frac{\sigma_f}{2\epsilon_o} & y > \frac{d}{2} \\ y < -\frac{d}{2} \end{cases}$$

$$P_y = (\epsilon - \epsilon_o)E_y = \begin{cases} \pm \frac{(\epsilon - \epsilon_o)\sigma_f}{2\epsilon} & 0 < y < \frac{d}{2} \\ -\frac{d}{2} < y < 0 \\ 0 & |y| > \frac{d}{2} \end{cases}$$

$$\sigma_p(y = \pm \frac{d}{2}) = -[P_y(y = \frac{d}{2}+) - P_y(y = \frac{d}{2}-)] = \frac{(\epsilon - \epsilon_o)\sigma_f}{2\epsilon}$$

$$\sigma_p(y=0) = -\frac{(\epsilon - \epsilon_o)\sigma_f}{\epsilon}$$

$$c) \quad D_r = \begin{cases} \frac{Qr}{4\pi R^3} \\ \frac{Q}{4\pi r^2} \end{cases}; \quad E_r = \frac{D_r}{\epsilon} = \begin{cases} \frac{Qr}{4\pi\epsilon R^3} & r < R \\ \frac{Q}{4\pi\epsilon_o r^2} & r > R \end{cases}$$

$$P_r = (\epsilon - \epsilon_o)E_r = \begin{cases} \frac{(\epsilon - \epsilon_o)Qr}{4\pi\epsilon R^3} & r < R \\ 0 & r > R \end{cases}$$

$$\sigma_p(r=R) = -[P_r(r=R_+) - P_r(r=R_-)] = \frac{(\epsilon - \epsilon_o)Q}{4\pi\epsilon R^2}$$

$$\rho_p(r) = -\nabla \cdot \bar{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P_r) = -\frac{(\epsilon - \epsilon_0)Q}{\frac{4}{3}\pi R^3 \epsilon}$$

$$18. \quad \bar{P} = P_o \bar{i}_z = P_o (\bar{i}_r \cos\theta - \bar{i}_\theta \sin\theta)$$

$$a) \quad \sigma_p = -P_r(r=R) = -P_o \cos\theta$$

$$b) \quad d\lambda_p = \sigma_p R d\theta = -P_o R \cos\theta d\theta, \quad dq = -2\pi P_o R^2 \sin\theta \cos\theta d\theta$$

$$c) \quad dE_z = -\frac{dq \cos\theta}{4\pi\epsilon_o R^2} = \frac{P_o}{2\epsilon_o} \cos^2\theta \sin\theta d\theta$$

$$d) \quad E_z = \frac{P_o}{2\epsilon_o} \int_0^\pi \cos^2\theta \sin\theta d\theta + E_o$$

$$\text{Let } u = \cos\theta, \quad du = -\sin\theta d\theta$$

$$E_z = -\frac{P_o}{2\epsilon_o} \int_1^{-1} u^2 du + E_o$$

$$= -\frac{P_o}{6\epsilon_o} u^3 \Big|_1^{-1} + E_o = \frac{P_o}{3\epsilon_o} + E_o$$

$$19. \quad a) \quad V_I = \frac{-1}{2\pi\epsilon_1} [\lambda \ln s_1 + \lambda' \ln r + \lambda'' \ln s_2]$$

$$V_{II} = \frac{-1}{2\pi\epsilon_2} \lambda''' \ln s_1$$

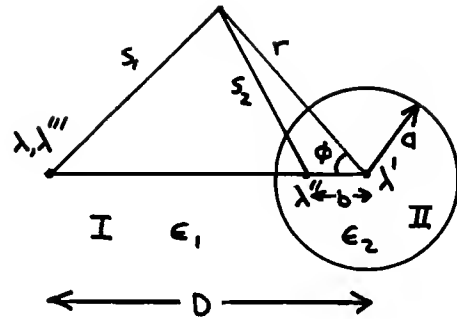
where

$$s_1 = [D^2 + r^2 - 2rD \cos\phi]^{1/2}$$

$$s_2 = [b^2 + r^2 - 2rb \cos\phi]^{1/2}; \quad b = \frac{a^2}{D}$$

$$\bar{E}_I = -\nabla V_I = \frac{1}{2\pi\epsilon_1} \left\{ \frac{\lambda}{s_1} [(r - D \cos\phi) \bar{i}_r + D \sin\phi \bar{i}_\phi] \right.$$

$$\left. + \frac{\lambda''}{s_2} [(r - b \cos\phi) \bar{i}_r + b \sin\phi \bar{i}_\phi] + \frac{\lambda'}{r} \bar{i}_r \right\}$$



$$\bar{E}_{II} = -\nabla V_{II} = \frac{\lambda'''}{2\pi\epsilon_2 s_1^2} [(r - D\cos\phi)\bar{i}_r + D\sin\phi\bar{i}_\phi]$$

At the cylinder surface the boundary conditions are

$$E_{\phi I}(r=a) = E_{\phi II}(r=a) \rightarrow \frac{1}{\epsilon_1} \left( \frac{\lambda D}{2} + \frac{\lambda'' b}{2} \right) = \frac{\lambda'''}{2\epsilon_2 s_1}$$

$$\epsilon_1 E_{rI}(r=a) = \epsilon_2 E_{rII}(r=a) \rightarrow \frac{\lambda'}{a} + \frac{\lambda}{2s_1} (a - D\cos\phi) + \frac{\lambda''}{2s_2} (a - b\cos\phi) = \frac{\lambda'''}{2s_1} (a - D\cos\phi)$$

With  $b = \frac{a^2}{D}$  at  $r = a$ ,  $s_2 = \frac{a}{D} s_1$  so that the boundary conditions reduce to

$$\left. \begin{aligned} \frac{(\lambda + \lambda'')}{\epsilon_1} &= \frac{\lambda'''}{\epsilon_2} \\ 2\lambda' + \lambda + \lambda'' &= \lambda''' \\ \lambda'(D^2 + a^2) + \lambda a^2 + \lambda'' D^2 &= \lambda''' a^2 \end{aligned} \right\} \rightarrow \begin{aligned} \lambda' &= -\lambda'' = \frac{\lambda(\epsilon_2 - \epsilon_1)}{\epsilon_1 + \epsilon_2} \\ \lambda''' &= \frac{2\epsilon_2 \lambda}{\epsilon_1 + \epsilon_2} \end{aligned}$$

b)  $\lim_{a \rightarrow \infty} b = \frac{a^2}{D} \approx a - d$   
 $D = d + a$

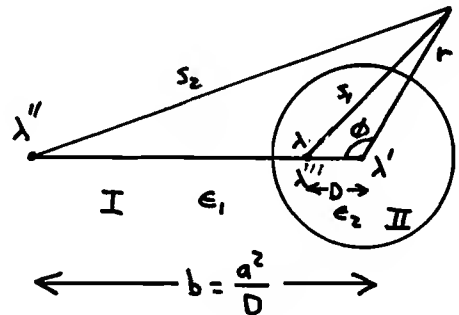
Then  $\lambda'$  is at infinity having no effect while  $\lambda''$  is now a distance  $d$  back from the surface as for a flat plane. The image charges  $\lambda''$  and  $\lambda'''$  depend on the permittivities in the same way as in (3.3.3).

c)  $F_x = \frac{-\lambda}{2\pi\epsilon_1} \left[ \frac{\lambda'}{D} + \frac{\lambda''}{D - b} \right]$   
 $= \frac{\lambda^2}{2\pi\epsilon_1} \frac{(\epsilon_2 - \epsilon_1)}{(\epsilon_1 + \epsilon_2)} \frac{a^2}{D(D^2 - a^2)}$

d)  $V_I = \frac{-1}{2\pi\epsilon_1} [\lambda' \ln r + \lambda''' \ln s_1]$

$$V_{II} = \frac{-1}{2\pi\epsilon_2} [\lambda \ln s_1 + \lambda'' \ln s_2]$$

$$\bar{E}_I = -\nabla V_I = \frac{1}{2\pi\epsilon_1} \left\{ \frac{\lambda'}{r} \bar{i}_r + \frac{\lambda'''}{2s_1} [(r - D\cos\phi)\bar{i}_r + D\sin\phi\bar{i}_\phi] \right\}$$



$$\bar{E}_{II} = -\nabla V_{II} = \frac{1}{2\pi\epsilon_2} \left\{ \frac{\lambda}{s_1} [(r - D\cos\phi)\bar{i}_r + D\sin\phi\bar{i}_\phi] + \frac{\lambda''}{s_2} [(r - b\cos\phi)\bar{i}_r + b\sin\phi\bar{i}_\phi] \right\}$$

$$\left. \begin{aligned} E_{\phi I}(r=a) &= E_{\phi II}(r=a) \\ \epsilon_1 E_{rI}(r=a) &= \epsilon_2 E_{rII}(r=a) \end{aligned} \right\} \rightarrow \begin{aligned} \lambda' &= \lambda'' = \frac{\lambda(\epsilon_2 - \epsilon_1)}{\epsilon_1 + \epsilon_2} \\ \lambda''' &= \frac{2\epsilon_1\lambda}{\epsilon_1 + \epsilon_2} \end{aligned}$$

$$\begin{aligned} F_x &= \frac{\lambda\lambda''}{2\pi\epsilon_2(D - \frac{a^2}{D})} \\ &= \frac{\lambda^2(\epsilon_2 - \epsilon_1)D}{2\pi\epsilon_2(\epsilon_1 + \epsilon_2)(D^2 - a^2)} \end{aligned}$$

20. a)  $\sigma_1 E_y(0_+) = \sigma_2 E_y(0_-)$   
 $E_x(0_+) = E_x(0_-)$

b) The solution is the same as in (3.3.3) if we replace the permittivity by the respective conductivities

$$q' = -\frac{(\sigma_2 - \sigma_1)q}{\sigma_1 + \sigma_2}, \quad q'' = \frac{2\sigma_2 q}{\sigma_1 + \sigma_2}$$

c)  $f_y = \frac{qq'}{4\pi\epsilon_1(2d)^2}$   
 $= \frac{-q^2(\sigma_2 - \sigma_1)}{16\pi\epsilon_1 d^2(\sigma_1 + \sigma_2)}$

21. a) When the electrodes are shorted  $E = 0$ ,  $D = P_o = \sigma_f \rightarrow q = P_o A$ .

b) When the upper electrode is moved to infinity,  $D \rightarrow 0$  so that all the charge flows off the electrode onto the capacitor.

$$v_c = \frac{q}{C} = \frac{P_o A}{C}$$

22. a)  $\sigma_p(z = \pm \frac{L}{2}) = \frac{P_o}{2}$



$$\rho_P(z) = -\nabla \cdot \vec{P} = -\frac{\partial P_z}{\partial z} = -\frac{P_0}{L}$$

- b) Superposing and spatially shifting the results of (2.3.5b) letting  $\sigma_0 = \frac{P_0}{2}$  and of (2.3.5d) letting  $\rho_0 = -P_0/L$  yields

$$D_z = \frac{P_0}{2} \left\{ \frac{-2(z - \frac{L}{2})}{[a^2 + (z - \frac{L}{2})^2]^{1/2}} - \frac{2(z + \frac{L}{2})}{[a^2 + (z + \frac{L}{2})^2]^{1/2}} - \frac{[a^2 + (z - \frac{L}{2})^2]^{1/2}}{L} + \frac{[a^2 + (z + \frac{L}{2})^2]^{1/2}}{L} \right\}$$

$$E_z = \begin{cases} \frac{D_z}{\epsilon_0} & |z| > \frac{L}{2} \\ \frac{D_z - P_0 z/L}{\epsilon_0} & |z| < \frac{L}{2} \end{cases}$$

23. a)  $\oint \vec{D} \cdot d\vec{S} = D_r 4\pi r^2 = 0 \Rightarrow D_r = 0$

$$D_r = \begin{cases} \epsilon_0 E_r + P_r & r < R \\ \epsilon_0 E_r & r > R \end{cases}$$

$$E_r = \begin{cases} \frac{-P_r}{\epsilon_0} = \frac{-P_0 r}{\epsilon_0 R} & r < R \\ 0 & r > R \end{cases}$$

b)  $E_0(s - b) + E_P b = V_0; \quad \epsilon_0 E_0 = \epsilon_0 E_P + P_0$

$$E_0 = \frac{V_0}{s} + \frac{P_0 b}{\epsilon_0 s}, \quad E_P = \frac{V_0}{s} - \frac{P_0(s - b)}{\epsilon_0 s}$$

24.  $E_b b + E_a a = v(t) \rightarrow E_a = \frac{v(t)}{a} - \frac{E_b b}{a}$

$$J_a = \sigma E_a, \quad \frac{\partial J_b}{\partial t} = \omega_{pe}^2 \epsilon_b E_b$$

$$J_a - J_b + \frac{\partial}{\partial t} [\epsilon_a E_a - \epsilon_b E_b] = 0$$

$$\frac{\partial^2 E_b}{\partial t^2} + \alpha \frac{\partial E_b}{\partial t} + \omega_o^2 E_b = \frac{\epsilon_a}{\epsilon_a b + \epsilon_b a} \frac{\partial^2 v}{\partial t^2} + \frac{\sigma}{\epsilon_a b + \epsilon_b a} \frac{\partial v}{\partial t}$$

$$\alpha = \frac{\sigma b}{\epsilon_a b + \epsilon_b a}, \quad \omega_o^2 = \frac{\omega_{pe}^2 \epsilon_b a}{\epsilon_a b + \epsilon_b a}$$

$$a) \quad E_b = \hat{E} e^{st}$$

$$s^2 + \alpha s + \omega_o^2 = 0 \rightarrow s = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \omega_o^2}$$

$$\text{Critically damped when } \omega_o^2 = \frac{\alpha^2}{4}$$

$$E_b = [\hat{E}_1 \sin \beta t + \hat{E}_2 \cos \beta t] e^{-\alpha t/2}; \quad \beta = \sqrt{\omega_o^2 - \frac{\alpha^2}{4}}$$

$$\frac{\partial E_b}{\partial t} = \{\beta [\hat{E}_1 \cos \beta t - \hat{E}_2 \sin \beta t] - \frac{\alpha}{2} [\hat{E}_1 \sin \beta t + \hat{E}_2 \cos \beta t]\} e^{-\alpha t/2}$$

Initial conditions:

$$E_b(t=0) = \frac{\epsilon_a V_o}{\epsilon_a b + \epsilon_b a} = \hat{E}_2$$

$$\left. \begin{aligned} \frac{\partial E_b}{\partial t} \Big|_{t=0} &= -\alpha E_b(t=0) + \frac{\sigma V_o}{\epsilon_a b + \epsilon_b a} = \beta \hat{E}_1 - \frac{\alpha \hat{E}_2}{2} \end{aligned} \right\} \rightarrow \hat{E}_1 = \frac{V_o [\sigma - \frac{\alpha \epsilon_a}{2}]}{\beta (\epsilon_a b + \epsilon_b a)}$$

$$\sigma_f = \epsilon_a E_a - \epsilon_b E_b$$

$$i(t) = \mathcal{L}D[\sigma E_a + \epsilon_a \frac{\partial E_a}{\partial t}] = \mathcal{L}D[J_b + \epsilon_b \frac{\partial E_b}{\partial t}]$$

$$b) \quad v(t) = \text{Re} V_o e^{j\omega t}, \quad E_b = \text{Re} \hat{E}_b e^{j\omega t}$$

$$\hat{E}_b = \frac{\frac{\epsilon_a V_o}{\epsilon_b a + \epsilon_a b} [-\omega^2 + \frac{\sigma}{\epsilon_a} j\omega]}{-\omega^2 + \alpha j\omega + \omega_o^2}, \quad \hat{E}_a = \frac{V_o}{a} - \frac{\hat{E}_b b}{a}$$

$$\hat{\sigma}_f = \epsilon_a \hat{E}_a - \epsilon_b \hat{E}_b, \quad \hat{i} = \ell D [\sigma + j\omega \epsilon_a] \hat{E}_a = \ell D \left[ \frac{\omega^2 \epsilon_b}{j\omega} + j\omega \epsilon_b \right] \hat{E}_b$$

## Section 3.4

25. a) Series

$$J = \sigma_1 E_1 = \sigma_2 E_2 = \frac{i}{\ell D}$$

$$E_1 b + E_2 a = V_o$$

$$\frac{i}{\ell D} \left[ \frac{b}{\sigma_1} + \frac{a}{\sigma_2} \right] = V_o$$

$$R = \frac{V_o}{i} = \left[ \frac{b}{\sigma_1} + \frac{a}{\sigma_2} \right] \frac{1}{\ell d}$$

$$b) \quad E_1 = \frac{A_1}{r}, \quad E_2 = \frac{A_2}{r}$$

$$A_1 \ell n \frac{R_2}{R_1} + A_2 \ell n \frac{R_3}{R_2} = V_o$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$A_2 = \frac{\sigma_1}{\sigma_2} A_1 = \frac{\sigma_1 V_o}{\left[ \sigma_1 \ell n \frac{R_3}{R_2} + \sigma_2 \ell n \frac{R_2}{R_1} \right]}$$

$$i = J_r 2\pi r \ell = \sigma_2 2\pi \ell A_2$$

Parallel

$$i = \sigma_1 E_1 a D + \sigma_2 E_2 b D$$

$$E_1 = E_2 = \frac{V_o}{s}$$

$$i = \frac{D V_o}{s} [\sigma_1 a + \sigma_2 b]$$

$$R = \frac{V_o}{i} = \frac{s}{D [\sigma_1 a + \sigma_2 b]}$$

$$E_r = \frac{V_o}{r \ell n \frac{R_2}{R_1}}$$

$$i = [\sigma_1 (2\pi - \alpha) + \sigma_2 \alpha] E_r r \ell$$

$$= \frac{V_o \ell}{\ell n \frac{R_2}{R_1}} [\sigma_1 (2\pi - \alpha) + \sigma_2 \alpha]$$

$$R = \frac{V_o}{i} = \frac{\ell n \frac{R_2}{R_1}}{\ell [\sigma_1 (2\pi - \alpha) + \sigma_2 \alpha]}$$

$$R = \frac{V_o}{i} = \frac{\sigma_1 \sigma_2 2\pi \ell V_o}{[\sigma_1 \ell n \frac{R_3}{R_2} + \sigma_2 \ell n \frac{R_2}{R_1}]}$$

$$R = \frac{V_o}{i} = \frac{\sigma_1 \ell n \frac{R_3}{R_2} + \sigma_2 \ell n \frac{R_2}{R_1}}{2\pi \ell \sigma_1 \sigma_2}$$

$$c) \quad E_1 = \frac{A_1}{r^2}, \quad E_2 = \frac{A_2}{r^2}$$

$$-A_1 \left( \frac{1}{R_2} - \frac{1}{R_1} \right) - A_2 \left( \frac{1}{R_3} - \frac{1}{R_2} \right) = V_o$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$A_2 = \frac{\sigma_1 A_1}{\sigma_2} = \frac{-V_o \sigma_1}{\sigma_1 \left( \frac{1}{R_3} - \frac{1}{R_2} \right) + \sigma_2 \left( \frac{1}{R_2} - \frac{1}{R_1} \right)}$$

$$i = J_r 4\pi r^2$$

$$= \sigma_2 E_2 4\pi r^2$$

$$= \sigma_2 A_2 4\pi$$

$$= \frac{4\pi \sigma_2 \sigma_1 V_o}{\sigma_1 \left( \frac{1}{R_2} - \frac{1}{R_3} \right) + \sigma_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right)}$$

$$R = \frac{V_o}{i}$$

$$= \frac{\sigma_1 \left( \frac{1}{R_2} - \frac{1}{R_3} \right) + \sigma_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right)}{4\pi \sigma_1 \sigma_2}$$

$$E_r = \frac{V_o}{\left( \frac{1}{R_1} - \frac{1}{R_2} \right) r^2}$$

$$i = 2\pi E_r^2 [\sigma_2 [1 - \cos \frac{\alpha}{2}] + \sigma_1 [\cos \frac{\alpha}{2} + 1]]$$

$$= \frac{2\pi V_o}{\left( \frac{1}{R_1} - \frac{1}{R_2} \right)} [\sigma_2 (1 - \cos \frac{\alpha}{2}) + \sigma_1 (\cos \frac{\alpha}{2} + 1)]$$

$$R = \frac{V_o}{i}$$

$$= \frac{\left( \frac{1}{R_1} - \frac{1}{R_2} \right)}{2\pi [\sigma_2 (1 - \cos \frac{\alpha}{2}) + \sigma_1 (1 + \cos \frac{\alpha}{2})]}$$

$$26. \quad a) \quad \sigma(x) E(x) = J_o \rightarrow E(x) = \frac{J_o}{\sigma_1 + (\sigma_2 - \sigma_1) \frac{x}{s}}$$

POLARIZATION AND CONDUCTION

$$\int_0^s E(x) dx = \frac{J_o s}{\sigma_2 - \sigma_1} \ln[\sigma_1 + (\sigma_2 - \sigma_1) \frac{x}{s}] \Big|_0^s = \frac{J_o s}{\sigma_2 - \sigma_1} \ln \frac{\sigma_2}{\sigma_1} = V_o$$

$$R = \frac{V_o}{i} = \frac{V_o}{J_o \ell D} = \frac{s}{\ell D (\sigma_2 - \sigma_1)} \ln \frac{\sigma_2}{\sigma_1}$$

$$b) \quad E(x) = \frac{V_o (\sigma_2 - \sigma_1)}{s \ln \frac{\sigma_2}{\sigma_1} [\sigma_1 + (\sigma_2 - \sigma_1) \frac{x}{s}]}$$

$$\rho_f = \epsilon \frac{dE}{dx} = - \frac{\epsilon V_o (\sigma_2 - \sigma_1)^2}{s^2 \ln \frac{\sigma_2}{\sigma_1} [\sigma_1 + (\sigma_2 - \sigma_1) \frac{x}{s}]^2}$$

$$\sigma_f(x=0) = \epsilon E(x=0) = \frac{\epsilon V_o (\sigma_2 - \sigma_1)}{s \sigma_1 \ln \frac{\sigma_2}{\sigma_1}}$$

$$\sigma_f(x=s) = -\epsilon E(x=s) = \frac{-\epsilon V_o (\sigma_2 - \sigma_1)}{s \sigma_2 \ln \frac{\sigma_2}{\sigma_1}}$$

$$c) \quad Q = \ell D \int_0^s \rho_f dx = \frac{-\epsilon V_o (\ell D) (\sigma_2 - \sigma_1)^2}{s^2 \ln \frac{\sigma_2}{\sigma_1}} \int_0^s \frac{dx}{[\sigma_1 + (\sigma_2 - \sigma_1) \frac{x}{s}]^2}$$

$$= \frac{\epsilon V_o \ell D (\sigma_2 - \sigma_1)}{s \ln \frac{\sigma_2}{\sigma_1} [\sigma_1 + (\sigma_2 - \sigma_1) \frac{x}{s}]} \Big|_0^s$$

$$= \frac{\epsilon V_o \ell D (\sigma_2 - \sigma_1)}{s \ln \frac{\sigma_2}{\sigma_1}} \left( \frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right)$$

$$= -\ell D [\sigma_f(x=0) + \sigma_f(x=s)]$$

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## 27. Rectangular

$$R_1 = \frac{[a+b]}{\sigma A}, R_2 = \frac{[a^2 + b^2]^{1/2}}{\sigma A} \rightarrow R = \frac{\frac{R_1}{2} R_2}{\frac{R_1}{2} + R_2} = \frac{R_1 R_2}{R_1 + 2R_2}$$

## Circular

$$R_1 = \frac{\pi R}{\sigma A}, R_2 = \frac{2R}{\sigma A} \rightarrow R = \frac{\frac{R_1}{2} R_2}{\frac{R_1}{2} + R_2} = \frac{R_1 R_2}{R_1 + 2R_2} = \frac{2\pi R}{\sigma A[\pi + 4]}$$

## Section 3.5

$$28. RC = \epsilon/\sigma \rightarrow R = \frac{\epsilon}{\sigma C}$$

$$C = \frac{2\pi\epsilon\ell}{\cosh^{-1}\left[\pm \frac{D^2 - R_1^2 - R_2^2}{2R_1 R_2}\right]} \rightarrow R = \frac{\cosh^{-1}\left[\pm \frac{D^2 - R_1^2 - R_2^2}{2R_1 R_2}\right]}{2\pi\sigma\ell}$$

## 29. Series

$$a) D = \epsilon_1 E_1 = \epsilon_2 E_2 = \sigma_f (x=0) = \frac{q}{\ell d}$$

$$E_1 b + E_2 a = V_o \rightarrow \frac{q}{\ell d} \left[ \frac{b}{\epsilon_1} + \frac{a}{\epsilon_2} \right] = V_o$$

$$C = \frac{q}{V_o} = \frac{\ell d}{\frac{b}{\epsilon_1} + \frac{a}{\epsilon_2}}$$

$$b) E_1 = \frac{A_1}{r}, E_2 = \frac{A_2}{r}$$

$$A_1 \ln \frac{R_2}{R_1} + A_2 \ln \frac{R_3}{R_2} = V_o$$

$$\epsilon_1 A_1 = \epsilon_2 A_2 \rightarrow A_2 = \frac{\epsilon_1}{\epsilon_2} A_1 = \frac{\epsilon_1 V_o}{\frac{R_3}{\epsilon_1 \ln \frac{R_2}{R_1}} + \frac{R_2}{\epsilon_2 \ln \frac{R_3}{R_1}}}$$

## Parallel

$$E = \frac{V_o}{s}$$

$$q = \frac{V_o d}{s} [\epsilon_1 a + \epsilon_2 b]$$

$$C = \frac{q}{V_o} = \frac{d}{s} (\epsilon_1 a + \epsilon_2 b)$$

$$E = \frac{V_o}{r \ln \frac{b}{a}}$$

$$q(R_1) = E(R_1) \ell R_1 [\epsilon_2 \alpha + \epsilon_1 (2\pi - \alpha)]$$

$$= \frac{V_o \ell}{\ln \frac{R_2}{R_1}} [\epsilon_2 \alpha + \epsilon_1 (2\pi - \alpha)]$$

# POLARIZATION AND CONDUCTION

$$q(R_1) = \epsilon_1 E_1(R_1) 2\pi R_1 \ell$$

$$= \epsilon_1 A_1 2\pi \ell$$

$$C = \frac{q(R_1)}{V_o} = \frac{\epsilon_1 \epsilon_2 2\pi \ell}{\epsilon_1 \ln \frac{R_3}{R_2} + \epsilon_2 \ln \frac{R_2}{R_1}}$$

$$c) \quad E_1 = \frac{A_1}{r^2}, \quad E_2 = \frac{A_2}{r^2}$$

$$A_1 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) + A_2 \left( \frac{1}{R_2} - \frac{1}{R_3} \right) = V_o$$

$$\epsilon_1 A_1 = \epsilon_2 A_2 = \frac{\epsilon_1 \epsilon_2 V_o}{\epsilon_1 \left( \frac{1}{R_2} - \frac{1}{R_3} \right) + \epsilon_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right)}$$

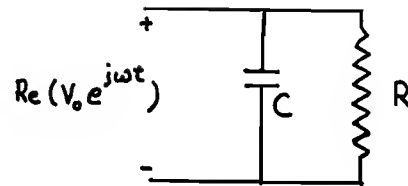
$$C = \frac{q(R_1)}{V_o} = \frac{4\pi \epsilon_1 \epsilon_2}{\epsilon_1 \left( \frac{1}{R_2} - \frac{1}{R_3} \right) + \epsilon_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right)}$$

$$C = \frac{q(R_1)}{V_o} = \frac{[\epsilon_2 \alpha + \epsilon_1 (2\pi - \alpha)] \ell}{\ln \frac{R_2}{R_1}}$$

$$E_r = \frac{V_o}{r^2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right)}$$

$$C = \frac{q(R_1)}{V_o} = \frac{2\pi}{\left( \frac{1}{R_1} - \frac{1}{R_2} \right)} [\epsilon_2 (1 - \cos \frac{\alpha}{2}) + \epsilon_1 (1 + \cos \frac{\alpha}{2})]$$

$$30. \quad R = \frac{V}{I}, \quad C = \frac{\epsilon}{\sigma R}$$



$$Y(j\omega) = Cj\omega + \frac{1}{R} = \frac{RCj\omega + 1}{R}$$

$$\hat{I} = V_o Y(j\omega) = \frac{V_o}{R} [RCj\omega + 1]$$

$$i(t) = \text{Re} \hat{I} e^{j\omega t} = \frac{V_o}{R} \cos \omega t - V_o C \omega \sin \omega t$$

$$31. \quad \nabla \cdot \vec{D} = \frac{1}{r} \frac{\partial}{\partial r} (r D_r) = 0 \rightarrow D_r = \frac{A}{r}$$

$$E_r = \frac{D_r}{\epsilon(r)} = \frac{A}{r \left[ \epsilon_1 + (\epsilon_2 - \epsilon_1) \frac{(r-a)}{b-a} \right]}$$

$$\begin{aligned} \int_a^b E_r dr = V_o &= \frac{-A}{\left[ \epsilon_1 - \frac{(\epsilon_2 - \epsilon_1)a}{b-a} \right]} \ln \left( \frac{\epsilon_1 + (\epsilon_2 - \epsilon_1) \frac{(r-a)}{(b-a)}}{r} \right) \Big|_{r=a}^b \\ &= \frac{-A(b-a)}{\epsilon_1 b - \epsilon_2 a} \ln \frac{\epsilon_2 a}{\epsilon_1 b} \end{aligned}$$

$$q(r=a) = D_r(r=a) 2\pi a l = 2\pi A l = - \frac{2\pi l (\epsilon_1 b - \epsilon_2 a) V_o}{(b-a) \ln \frac{\epsilon_2 a}{\epsilon_1 b}}$$

$$C = \frac{q(r=a)}{V_o} = \frac{2\pi l (\epsilon_2 a - \epsilon_1 b)}{(b-a) \ln \frac{\epsilon_2 a}{\epsilon_1 b}}$$

### Section 3.6

$$32. \quad a) \quad \rho_f(t) = \rho_o e^{-t/\tau}, \quad E(x) = \begin{cases} \frac{\rho_f(t)x}{\epsilon_o} + C_1 & 0 \leq x \leq b \\ C_2 & b \leq x \leq a+b \end{cases}; \quad \tau = \epsilon_o / \sigma$$

$$\int_0^{a+b} E(x) dx = \frac{\rho_f b^2}{2\epsilon_o} + C_1 b + C_2 a = V_o$$

$$\bar{n} \cdot [\bar{J}_a - \bar{J}_b] + \frac{\partial \sigma_f}{\partial t} = 0 \rightarrow -\sigma \left[ \frac{\rho_f(t)b}{\epsilon_o} + C_1 \right] + \frac{\partial}{\partial t} \left[ \epsilon_o \left[ C_2 - C_1 - \frac{\rho_f(t)b}{\epsilon_o} \right] \right] = 0$$

$$\frac{\partial C_2}{\partial t} + \frac{C_2 a}{(a+b)\tau} = \frac{V_o}{(a+b)\tau} \rightarrow C_2(t) = A e^{-\frac{a}{a+b} \frac{t}{\tau}} + \frac{V_o}{a}$$

$$\sigma_f(t=0) = \epsilon_o \left( C_2 - \frac{\rho_f(t=0)b}{\epsilon_o} - C_1 \right) = 0 \rightarrow A + \frac{V_o}{a} - \frac{\rho_o b}{\epsilon_o} - C_1 = 0$$

$$A = \frac{\rho_o b^2}{2\epsilon_o(a+b)} - \frac{V_o b}{a(a+b)}$$



$$C_2(t) = \left[ \frac{\rho_o b^2}{2\epsilon_o(a+b)} - \frac{V_o b}{a(a+b)} \right] e^{-\frac{a}{a+b} \frac{t}{\tau}} + \frac{V_o}{a}$$

$$C_1(t) = \frac{V_o}{b} - \frac{\rho_f(t)b}{2\epsilon_o} - \frac{C_2 a}{b} = \frac{-\rho_o b e^{-t/\tau}}{2\epsilon_o} - \left[ \frac{\rho_o ab}{2\epsilon_o(a+b)} - \frac{V_o}{a+b} \right] e^{-\frac{a}{a+b} \frac{t}{\tau}}$$

$$\begin{aligned} \text{b) } \sigma_f(x=b) &= \epsilon_o [E(x=b_+) - E(x=b_-)] \\ &= \epsilon_o \left[ C_2 - \frac{\rho_f(t)b}{\epsilon_o} - C_1 \right] \\ &= \epsilon_o \left[ \frac{V_o}{a} + \left( \frac{\rho_o b}{2\epsilon_o} - \frac{V_o}{a} \right) e^{-\frac{a}{a+b} \frac{t}{\tau}} - \frac{\rho_o b}{2\epsilon_o} e^{-t/\tau} \right] \end{aligned}$$

$$\begin{aligned} \text{c) } f_x &= \int_0^{b_-} \rho_f(x) E dx + \frac{1}{2} \sigma_f(x=b) [E(x=b_+) + E(x=b_-)] \\ &= \int_0^{b_-} \epsilon_o E \frac{\partial E}{\partial x} dx + \frac{\epsilon_o}{2} [E(x=b_+) - E(x=b_-)] [E(x=b_+) + E(x=b_-)] \\ &= \frac{1}{2} \epsilon_o [E^2(x=b_-) - E^2(x=0_+)] + \frac{\epsilon_o}{2} [E^2(x=b_+) - E^2(x=b_-)] \\ &= \frac{1}{2} \epsilon_o [E^2(x=b_+) - E^2(x=0_+)] \\ &= \frac{1}{2} \epsilon_o [C_2^2 - C_1^2] \end{aligned}$$

33.

$$E_r = \begin{cases} \frac{\rho_o r^2}{3\epsilon a_o} e^{-t/\tau} & 0 < r < a_o \\ \frac{\rho_o a_o^2}{3\epsilon r} e^{-t/\tau} & a_o < r < a_1 ; \quad \tau = \epsilon/\sigma \\ \frac{\rho_o a_o^2}{3\epsilon_o r} & r > a_1 \end{cases}$$

$$\sigma_f = \epsilon_o E_r(r=a_{1+}) - \epsilon E_r(r=a_{1-})$$

$$= \frac{\rho_o a_o^2}{3a_1} (1 - e^{-t/\tau})$$

$$34. \quad a) \quad E_r = \begin{cases} \frac{C_1}{r} & R_1 < r < R_2 \\ \frac{C_2}{r} & R_2 < r < R_3 \end{cases} \rightarrow \int_{R_1}^{R_3} E_r dr = C_1 \ln \frac{R_2}{R_1} + C_2 \ln \frac{R_3}{R_2} = V_o$$

$$\text{At } t = 0, \sigma_f(r=R_2) = 0$$

$$\epsilon_1 C_1 = \epsilon_2 C_2 = \frac{\epsilon_1 \epsilon_2 V_o}{\epsilon_1 \ln \frac{R_3}{R_2} + \epsilon_2 \ln \frac{R_2}{R_1}}$$

$$b) \quad J_r(r=R_{2+}) = J_r(r=R_{2-}) \rightarrow \sigma_1 C_1 = \sigma_2 C_2 = \frac{\sigma_1 \sigma_2 V_o}{\sigma_1 \ln \frac{R_3}{R_2} + \sigma_2 \ln \frac{R_2}{R_1}}$$

$$c) \quad J_r(r=R_{2+}) - J_r(r=R_{2-}) + \frac{\partial \sigma_f}{\partial t} = 0$$

$$\sigma_2 C_2 - \sigma_1 C_1 + \frac{\partial}{\partial t} [\epsilon_2 C_2 - \epsilon_1 C_1] = 0$$

$$\frac{\partial C_1}{\partial t} + \frac{C_1}{\tau} = \frac{\sigma_2 V_o}{\epsilon_1 \ln \frac{R_3}{R_2} + \epsilon_2 \ln \frac{R_2}{R_1}}; \quad \tau = \frac{\epsilon_1 \ln \frac{R_3}{R_2} + \epsilon_2 \ln \frac{R_2}{R_1}}{\sigma_1 \ln \frac{R_3}{R_2} + \sigma_2 \ln \frac{R_2}{R_1}}$$

$$C_1 = \frac{\sigma_2 V_o [1 - e^{-t/\tau}]}{\sigma_1 \ln \frac{R_3}{R_2} + \sigma_2 \ln \frac{R_2}{R_1}} + \frac{\epsilon_2 V_o e^{-t/\tau}}{\epsilon_1 \ln \frac{R_3}{R_2} + \epsilon_2 \ln \frac{R_2}{R_1}}$$

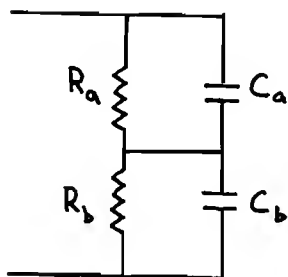
$$C_2 = \frac{V_o - C_1 \ln \frac{R_2}{R_1}}{\ln \frac{R_3}{R_2}} = \frac{\sigma_1 V_o}{\sigma_1 \ln \frac{R_3}{R_2} + \sigma_2 \ln \frac{R_2}{R_1}} + \frac{V_o \ln \frac{R_2}{R_1}}{\ln \frac{R_3}{R_2}} \left[ \frac{\epsilon_2}{\epsilon_1 \ln \frac{R_3}{R_2} + \epsilon_2 \ln \frac{R_2}{R_1}} - \frac{\sigma_2}{\sigma_1 \ln \frac{R_3}{R_2} + \sigma_2 \ln \frac{R_2}{R_1}} \right] e^{-t/\tau}$$

$$\sigma_f(r=b) = \frac{\epsilon_2 C_2 - \epsilon_1 C_1}{b}$$

$$d) \quad E_r = \begin{cases} \operatorname{Re} \frac{\hat{C}_1 e^{j\omega t}}{r} & R_1 < r < R_2 \\ \operatorname{Re} \frac{\hat{C}_2 e^{j\omega t}}{r} & R_2 < r < R_3 \end{cases} \rightarrow \hat{C}_1 \ln \frac{R_2}{R_1} + \hat{C}_2 \ln \frac{R_3}{R_2} = V_o$$

$$(\sigma_2 + j\omega\epsilon_2)\hat{C}_2 = (\sigma_1 + j\omega\epsilon_1)\hat{C}_1 = \frac{(\sigma_1 + j\omega\epsilon_1)(\sigma_2 + j\omega\epsilon_2)V_o}{(j\omega\epsilon_1 + \sigma_1) \ln \frac{R_3}{R_2} + (j\omega\epsilon_2 + \sigma_2) \ln \frac{R_2}{R_1}}$$

e)



$$R_a = \frac{\ln \frac{R_2}{R_1}}{2\pi\sigma_1 \ell}, \quad R_b = \frac{\ln \frac{R_3}{R_2}}{2\pi\sigma_2 \ell}$$

$$C_a = \frac{2\pi\epsilon_1 \ell}{\ln \frac{R_2}{R_1}}, \quad C_b = \frac{2\pi\epsilon_2 \ell}{\ln \frac{R_3}{R_2}}$$

$$35. \quad \nabla \cdot (\rho_f \bar{U}) + \frac{\sigma}{\epsilon} \rho_f = 0 \rightarrow \rho_f \nabla \cdot \bar{U} + (\bar{U} \cdot \nabla) \rho_f + \frac{\sigma}{\epsilon} \rho_f = 0$$

$$\bar{U} = \frac{A}{r^2} \bar{i}_r \rightarrow \nabla \cdot \bar{U} = 0 \rightarrow \frac{A}{r^2} \frac{d\rho_f}{dr} + \frac{\sigma}{\epsilon} \rho_f = 0$$

$$\frac{d\rho_f}{\rho_f} = \frac{-\sigma}{\epsilon A} r^2 dr \rightarrow \ln \rho_f = \frac{-\sigma r^3}{3\epsilon A} + C$$

$$\rho_f = \rho_o e^{-\sigma r^3 / 3\epsilon A}$$

$$\nabla \cdot \bar{E} = \frac{\rho_f}{\epsilon} \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = \frac{\rho_o}{\epsilon} e^{-\sigma r^3 / 3\epsilon A}$$

$$E_r = \frac{-\rho_o A}{\sigma r^2} e^{-\sigma r^3 / 3\epsilon A} + \frac{C_1}{r^2}$$

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$$E_r(r=0) \text{ finite} \rightarrow E_r = \frac{\rho_o A}{\sigma r^2} (1 - e^{-\sigma r^3/3\epsilon A})$$

36. a)  $\rho_f(x) = \rho_o e^{-x/\ell_m} = \epsilon \frac{dE}{dx}; \quad \ell_m = \frac{\epsilon U}{\sigma}$

$$E = \frac{-\rho_o \ell_m}{\epsilon} e^{-x/\ell_m} + C$$

$$\int_0^{\ell} E dx = C\ell + \frac{\rho_o \ell_m^2}{\epsilon} (e^{-\ell/\ell_m} - 1) = V_o$$

$$E = \frac{-\rho_o \ell_m}{\epsilon} e^{-x/\ell_m} + \frac{V_o}{\ell} - \frac{\rho_o \ell_m^2}{\epsilon \ell} (e^{-\ell/\ell_m} - 1)$$

b)  $f_x = \int_0^{\ell} \rho_f E dx$

$$= \int_0^{\ell} A \epsilon E \frac{dE}{dx} dx$$

$$= \frac{1}{2} \epsilon A [E^2(\ell) - E^2(0)]$$

c)  $\frac{i}{A} = \sigma E + \rho_f U \rightarrow E = \frac{i}{A\sigma} - \frac{\rho_o U}{\sigma} e^{-x/\ell_m}$

$$\int_0^{\ell} E dx = \frac{i\ell}{A\sigma} + \frac{\rho_o U \ell_m}{\sigma} (e^{-\ell/\ell_m} - 1) = -iR_L$$

$$i = \frac{\rho_o U \ell_m}{\sigma} \frac{(1 - e^{-\ell/\ell_m})}{R_L + \frac{\ell}{A\sigma}}$$

37.  $v(z, t=0_-) = V_o \frac{\cosh \sqrt{2RG} (z - \ell)}{\cosh \sqrt{2RG} \ell}; \quad i(z, t=0_-) = V_o \sqrt{\frac{G}{2R}} \frac{\sinh \sqrt{2RG} (z - \ell)}{\cosh \sqrt{2RG} \ell}$

If the  $z = 0$  end is either open or short circuited, the steady state voltage and current distributions are zero. The transient voltage and current are of the form

$$v(z, t) = \hat{V}(z) e^{-\alpha t}, \quad i(z, t) = \hat{I}(z) e^{-\alpha t}$$

with

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$$\hat{V}(z) = V_1 \sin pz + V_2 \cos pz$$

$$\hat{I}(z) = -\frac{1}{2R} \frac{d\hat{V}}{dz} = \frac{-p}{2R} [V_1 \cos pz - V_2 \sin pz]; \quad p^2 = 2RC\alpha - 2RG$$

Because the line is open circuited at  $z = \ell$ ,

$$\hat{i}(z=\ell) = 0 \rightarrow V_1 \cos p\ell - V_2 \sin p\ell = 0$$

a) Open circuit at  $z = 0$

$$\hat{I}(z=0) = 0 = V_1 \rightarrow \sin p\ell = 0 \rightarrow p\ell = n\pi$$

$$i(z,t) = \sum_{n=1}^{\infty} I_n \sin \frac{n\pi z}{\ell} e^{-\alpha_n t}; \quad \alpha_n = \frac{(\frac{n\pi}{\ell})^2 + 2RG}{2RC}$$

Initial conditions:

$$i(z,t=0) = V_o \sqrt{\frac{G}{2R}} \frac{\sinh \sqrt{2RG} (z - \ell)}{\cosh \sqrt{2RG} \ell} = \sum_{n=1}^{\infty} I_n \sin \frac{n\pi z}{\ell}$$

$$I_n = -\frac{2V_o \sqrt{\frac{G}{2R}} n\pi \tanh \sqrt{2RG} \ell}{\ell^2 [(\frac{n\pi}{\ell})^2 + 2RG]}$$

$$v(z,t) = \sum_{n=1}^{\infty} \frac{2RI_n}{(n\pi/\ell)} \cos \frac{n\pi z}{\ell} e^{-\alpha_n t}$$

b) Short circuit at  $z = 0$

$$\hat{V}(z=0) = 0 = V_2 \rightarrow \hat{V}(z) = V_1 \sin pz$$

$$\hat{I}(z) = -\frac{1}{2R} \frac{d\hat{V}}{dz} = -\frac{pV_1}{2R} \cos pz$$

$$\hat{I}(z=\ell) = 0 \rightarrow p\ell = (2n+1) \frac{\pi}{2}, \quad n = 0, 1, 2, \dots$$

Then

$$v(z,t) = \sum_{n=0}^{\infty} V_n \sin \frac{(2n+1)\pi z}{2\ell} e^{-\alpha_n t}$$

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$$v(z, t=0) = V_0 \frac{\cosh \sqrt{2RG} (z - \ell)}{\cosh \sqrt{2RG} \ell} = \sum_{n=0}^{\infty} V_n \sin \frac{(2n+1)\pi z}{2\ell}$$

$$V_n = \frac{V_0 \pi (2n+1)}{\ell^2 [2RG + \left(\frac{(2n+1)\pi}{2\ell}\right)^2]}$$

$$i(z, t) = -\frac{1}{2R} \frac{\partial v}{\partial z} = \frac{-\pi}{4R\ell} \sum_{n=0}^{\infty} (2n+1) V_n \cos(2n+1) \frac{\pi z}{2\ell} e^{-\alpha_n t}$$

38. a)  $v(z) = V_1 \sinh \sqrt{2RG} z + V_2 \cosh \sqrt{2RG} z$

$$v(z=0) = V_0 \rightarrow v(z) = -\frac{V_0 \sinh \sqrt{2RG} (z - \ell)}{\sinh \sqrt{2RG} \ell}$$

$$v(z=\ell) = 0$$

$$i(z) = -\frac{1}{2R} \frac{dv}{dz} = V_0 \sqrt{\frac{G}{2R}} \frac{\cosh \sqrt{2RG} (z - \ell)}{\sinh \sqrt{2RG} \ell}$$

b)  $v(z, t) = -\frac{V_0 \sinh \sqrt{2RG} (z - \ell)}{\sinh \sqrt{2RG} \ell} + \hat{v}(z) e^{-\alpha t}$

$$\hat{v}(z) = V_1 \sin p z + V_2 \cos p z, \quad p^2 = 2RC\alpha - 2RG$$

$$\hat{v}(z=0) = 0$$

$$\rightarrow \hat{v}(z) = \sum_{n=1}^{\infty} V_n \sin \frac{n\pi z}{\ell}, \quad p = \frac{n\pi}{\ell}$$

$$\hat{v}(z=\ell) = 0$$

$$v(z, t) = -\frac{V_0 \sinh \sqrt{2RG} (z - \ell)}{\sinh \sqrt{2RG} \ell} + \sum_{n=1}^{\infty} V_n \sin \frac{n\pi z}{\ell} e^{-\alpha_n t}$$

$$v(z, t=0) = 0 \rightarrow V_n = \frac{-2n\pi V_0}{\ell^2 [2RG + \left(\frac{n\pi}{\ell}\right)^2]}$$

$$i(z, t) = -\frac{1}{2R} \frac{\partial v}{\partial z} = V_0 \sqrt{\frac{G}{2R}} \frac{\cosh \sqrt{2RG} (z - \ell)}{\sinh \sqrt{2RG} \ell} - \frac{1}{2R} \sum_{n=1}^{\infty} \frac{n\pi}{\ell} V_n \cos \frac{n\pi z}{\ell} e^{-\alpha_n t}$$

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$$39. \quad -\frac{\partial v}{\partial z} = 2iR \quad \rightarrow \quad \frac{\partial^2 v}{\partial z^2} = 2RC \frac{\partial v}{\partial t} + 2RGv$$

$$-\frac{\partial i}{\partial z} = C \frac{\partial v}{\partial t} + Gv$$

$$v(z,t) = \text{Re } \hat{v}(z)e^{j\omega t}$$

$$\frac{d^2 \hat{v}}{dz^2} - 2R[Cj\omega + G]\hat{v} = 0 \rightarrow \hat{v}(z) = V_1 \sin pz + V_2 \cosh pz$$

$$p^2 = -2R(G + Cj\omega)$$

a) Short circuited at  $z = \ell$

$$\begin{aligned} \hat{v}(z=\ell) &= 0 \\ \hat{v}(z=0) &= V_o \end{aligned} \quad \rightarrow \quad \hat{v}(z) = -\frac{V_o \sinh p(z - \ell)}{\sinh p\ell}$$

$$\hat{i}(z) = -\frac{1}{2R} \frac{d\hat{v}}{dz} = \frac{p}{2R} V_o \frac{\cosh p(z - \ell)}{\sinh p\ell}$$

Open circuited at  $z = \ell$

$$\hat{i}(z) = -\frac{p}{2R} [V_1 \cosh pz - V_2 \sinh pz]$$

$$\begin{aligned} \hat{v}(z=0) &= V_o \\ \hat{i}(z=\ell) &= 0 \end{aligned} \quad \rightarrow \quad \hat{v}(z) = V_o \frac{\cosh p(z - \ell)}{\cosh p\ell}$$

$$\hat{i}(z) = -\frac{pV_o}{2R} \frac{\sinh p(z - \ell)}{\cosh p\ell}$$

b) Short Circuit

$$p\ell = n\pi \rightarrow j\omega = -\frac{(\frac{n\pi}{\ell})^2}{2RC} - \frac{G}{C}$$

Open Circuit

$$p\ell = (2n + 1) \frac{\pi}{2} \rightarrow j\omega = -\frac{[(2n + 1)\frac{\pi}{2\ell}]^2}{2RC} - \frac{G}{C}$$

$$c) \quad \langle P \rangle = \frac{1}{2} \text{Re}[\hat{v}(z=0)\hat{i}^*(z=0)]$$

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Short circuit:

$$\langle P \rangle = \frac{V^2}{4R} \operatorname{Re}[p^* \cot p^* \ell]$$

Open circuit:

$$\langle P \rangle = \frac{V^2}{4R} \operatorname{Re}[p^* \tanh p^* \ell]$$

$$40. \quad R_{\text{inner}} = \frac{\ell}{\sigma_c \pi a^2}, \quad R_{\text{outer}} = \frac{\ell}{\sigma_c \pi (c^2 - b^2)} \rightarrow 2R = R_{\text{inner}} + R_{\text{outer}} = \frac{\ell}{\sigma_c \pi} \left[ \frac{1}{a^2} + \frac{1}{c^2 - b^2} \right]$$

$$C = \frac{2\pi\epsilon\ell}{\ln \frac{b}{a}}, \quad G = \frac{2\pi\sigma\ell}{\ln \frac{b}{a}}$$

## Section 3.7

$$41. \quad a) \quad \nabla \cdot \bar{J} + \frac{\partial \rho_f}{\partial t} = 0; \quad \bar{J} = \rho_f \bar{uE}$$

$$\rho_f = \epsilon \nabla \cdot \bar{E} \rightarrow \rho_f = \epsilon \frac{\partial E}{\partial x}$$

$$\nabla \cdot [\rho_f \bar{uE} + \epsilon \frac{\partial \bar{E}}{\partial t}] = 0 \rightarrow \frac{\partial}{\partial x} [\epsilon \mu E \frac{\partial E}{\partial x} + \epsilon \frac{\partial E}{\partial t}] = 0$$

$$\underbrace{\epsilon \mu E \frac{\partial E}{\partial x}}_{\substack{\text{conduction} \\ \text{current} \\ \text{density}}} + \underbrace{\epsilon \frac{\partial E}{\partial t}}_{\substack{\text{displacement} \\ \text{current} \\ \text{density}}} = \underbrace{J(t)}_{\substack{\text{total} \\ \text{current} \\ \text{density}}}$$

$$b) \quad \frac{\epsilon \mu}{2} [E^2(\ell) - E^2(0)] + \epsilon \frac{dv}{dt} = J(t) \ell$$

$$c) \quad \text{Ahead of the front } \rho_f = 0 \rightarrow J(t) = \epsilon \frac{dE(\ell)}{dt} = \frac{\epsilon \mu}{2\ell} E^2(\ell)$$

where  $E(0) = 0$  (SCL) and  $\frac{dv}{dt} = 0$  for  $t > 0$

$$\frac{dE(\ell)}{E^2(\ell)} = \frac{\mu dt}{2\ell} \rightarrow -\frac{1}{E(\ell)} = \frac{\mu t}{2\ell} + C$$



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$$E(\ell, t=0) = \frac{V_o}{\ell} \rightarrow E(\ell) = \frac{V_o/\ell}{1 - \frac{\mu t V_o}{2\ell^2}}$$

d)  $E(x=s, t) = E(\ell, t)$

e)  $\frac{ds}{dt} = \mu E(s, t) = \frac{\mu V_o/\ell}{1 - \frac{\mu t V_o}{2\ell^2}} \rightarrow s = -2\ell \ln[1 - \frac{\mu V_o t}{2\ell^2}]$

f)  $s = \ell = -2\ell \ln[1 - \frac{\mu V_o \tau}{2\ell^2}] \rightarrow \tau = \frac{2\ell^2}{\mu V_o} (1 - e^{-1/2})$

g), h) Steady state:

$$\epsilon \mu E \frac{\partial E}{\partial x} = J_o \rightarrow E = \sqrt{\frac{2J_o x}{\epsilon \mu} + E_o^2}$$

$$\rho_f = \frac{J_o}{\mu E} = \frac{J_o}{\mu \sqrt{\frac{2J_o x}{\epsilon \mu} + E_o^2}}$$

$$E = -\frac{dV}{dx} \rightarrow V = -\frac{\epsilon \mu}{3J_o} \left\{ \left[ \frac{2J_o x}{\epsilon \mu} + E_o^2 \right]^{3/2} - E_o^3 \right\}$$

where  $V(x=0) = 0$

$$V(x=\ell) = -V_o = -\frac{\epsilon \mu}{3J_o} \left\{ \left[ \frac{2J_o \ell}{\epsilon \mu} + E_o^2 \right]^{3/2} - E_o^3 \right\}$$

$$J_o = \frac{\epsilon \mu V_o^2}{16\ell^3} \left\{ 9 - 12 \frac{E_o \ell}{V_o} + \left[ (9 - 12 \frac{E_o \ell}{V_o})^2 + 192(1 - \frac{E_o \ell}{V_o}) (\frac{E_o \ell}{V_o})^3 \right]^{1/2} \right\}$$

$$SCL \rightarrow E_o = 0 \rightarrow J_o = \frac{9}{8} \frac{\epsilon \mu V_o^2}{\ell^3}$$

42. a)  $\rho_f \mu E_r = \frac{I}{2\pi r \ell} ; \quad \nabla \cdot \vec{E} = \frac{1}{r} \frac{\partial}{\partial r} (r E_r) = \frac{\rho_f}{\epsilon}$

$$E_r \frac{\partial}{\partial r} (r E_r) = \frac{I}{2\pi \ell \epsilon \mu} \rightarrow (r E_r)^2 = \frac{I}{2\pi \ell \epsilon \mu} (r^2 - R_i^2) + R_i^2 E_i^2$$

$$E_r = \frac{1}{r} \left[ \frac{I}{2\pi \ell \epsilon \mu} (r^2 - R_i^2) + R_i^2 E_i^2 \right]^{1/2}; \quad \rho_f = \frac{I}{2\pi r \ell \mu E_r}$$

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$$b) \quad E_i = 0 \rightarrow E_r = \frac{1}{r} \left[ \frac{I}{2\pi\epsilon\mu\ell} \right]^{1/2} [r^2 - R_i^2]^{1/2}$$

$$\int_{R_i}^{R_o} E_r dr = \left[ \frac{I}{2\pi\epsilon\mu\ell} \right]^{1/2} \left\{ \sqrt{r^2 - R_i^2} - R_i \cos^{-1} \frac{R_i}{r} \right\} \bigg|_{r=R_i}^{R_o} = V_o$$

$$I = \frac{2\pi\epsilon\mu\ell V_o^2}{\left\{ \sqrt{R_o^2 - R_i^2} - R_i \cos^{-1} \frac{R_i}{R_o} \right\}^2}$$

$$c) \quad E_i^2 = \frac{I}{2\pi\epsilon\mu\ell} \rightarrow E(r) = E_i \rightarrow \int_{R_i}^{R_o} E(r) dr = E_i (R_o - R_i) = V_o$$

$$E_i^2 = \left( \frac{V_o}{R_o - R_i} \right)^2 = \frac{I}{2\pi\epsilon\mu\ell} \rightarrow I = 2\pi\epsilon\mu\ell \left( \frac{V_o}{R_o - R_i} \right)^2$$

$$d) \quad \rho_f \mu E_r = \frac{I}{4\pi r^2}; \quad \nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = \frac{\rho_f}{\epsilon}$$

$$E_r \frac{\partial}{\partial r} (r^2 E_r) = \frac{I}{4\pi\epsilon\mu} \rightarrow \frac{(r^2 E_r)^2}{2} = \frac{I(r^3 - R_i^3)}{12\pi\epsilon\mu} + \frac{(R_i^2 E_i)^2}{2}$$

$$E_r = \left( \frac{I}{6\pi\epsilon\mu} \right)^{1/2} \frac{1}{r^2} [r^3 - R_i^3 + \frac{R_i^4 E_i^2}{I} 6\pi\epsilon\mu]^{1/2}$$

$$\rho_f = \frac{I}{4\pi r^2 \mu E_r}$$

Section 3.8

$$43. \quad a) \quad W = \frac{1}{2} q [V(\vec{r} + \vec{d}) - V(\vec{r})]$$

$$\approx \frac{1}{2} q \nabla V \cdot \vec{d}$$

$$\approx - \frac{1}{2} \vec{p} \cdot \vec{E}$$

$$b) \quad W = - \frac{1}{2} p \frac{(2p)(2)}{4\pi\epsilon_o s^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \approx - \frac{1.2 p^2}{2\pi\epsilon_o s^3}$$

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$$c) W = -\frac{1}{2} \frac{p(2p)(2)}{4\pi\epsilon_o s^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \approx \frac{.9 p^2}{2\pi\epsilon_o s^3}$$

$$d) W = -\frac{1}{2} p \left( \frac{-p}{4\pi\epsilon_o s^3} \right)^{(2)} \sum_{n=1}^{\infty} \frac{1}{n^3} \approx \frac{1.2 p^2}{4\pi\epsilon_o s^3} \quad (\text{identical})$$

$$W = -\frac{1}{2} p \left( \frac{-p}{4\pi\epsilon_o s^3} \right)^{(2)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \approx \frac{-.9 p^2}{4\pi\epsilon_o s^3} \quad (\text{alternating polarity})$$

$$44. \quad \bar{E} = \frac{p}{4\pi\epsilon_o r^3} [2\cos\theta \bar{i}_r + \sin\theta \bar{i}_\theta] \rightarrow \frac{1}{2} \epsilon_o |\bar{E}|^2 = \frac{p^2}{32\pi^2 \epsilon_o r^6} [4\cos^2\theta + \sin^2\theta]$$

$$\begin{aligned} W &= \int_{\theta=0}^{\pi} \int_{r=R}^{\infty} \frac{1}{2} \epsilon_o |\bar{E}|^2 2\pi r^2 \sin\theta dr d\theta \\ &= \frac{p^2}{16\pi\epsilon_o} \int_{\theta=0}^{\pi} \int_{r=R}^{\infty} \frac{[4\cos^2\theta + \sin^2\theta]}{r^4} \sin\theta dr d\theta \\ &= \frac{p^2}{48\pi\epsilon_o R^3} \int_{\theta=0}^{\pi} [4\cos^2\theta + \sin^2\theta] \sin\theta d\theta \\ &= \frac{p^2}{48\pi\epsilon_o R^3} \left[ -\frac{4\cos^3\theta}{3} - \frac{1}{3} \cos\theta (\sin^2\theta + 2) \right] \Big|_{\theta=0}^{\pi} \\ &= \frac{p^2}{12\pi\epsilon_o R^3} \end{aligned}$$

45. a) Each drop carries a charge  $\frac{Q}{N}$  with radius  $r$  so that the volume is conserved

$$N \frac{4}{3} \pi r^3 = \frac{4}{3} \pi R^3 \rightarrow r = \frac{R}{N^{1/3}}$$

$$b) W_e = \frac{N \left( \frac{Q}{N} \right)^2}{8\pi\epsilon_o (R/N^{1/3})} = \frac{Q^2}{8\pi\epsilon_o R N^{2/3}}$$

$$c) W_{\text{final}} = N W_s 4\pi \left( \frac{R}{N^{1/3}} \right)^2 + W_e = 4\pi R^2 W_s N^{1/3} + \frac{Q^2}{8\pi\epsilon_o R N^{2/3}}$$

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$$d) W_{\text{original}} = W_S 4\pi R^2 + \frac{Q^2}{8\pi\epsilon_0 R}$$

$$\text{Work} = W_{\text{final}} - W_{\text{original}} = W_S 4\pi R^2 [N^{1/3} - 1] + \frac{Q^2}{8\pi\epsilon_0 R} (N^{-2/3} - 1)$$

$$e) \frac{\partial \text{Work}}{\partial N} = \frac{W_S 4\pi R^2}{3N^{2/3}} - \frac{2}{3} \frac{Q^2}{8\pi\epsilon_0 R N^{5/3}} = 0 \rightarrow N = \frac{Q^2}{16\pi^2 \epsilon_0 R^2 W_S}$$

$$N = \frac{10^{-12}}{16\pi^2 (8.854 \times 10^{-12}) (10^{-6}) (.072)} \approx 10^4 \text{ droplets of radius } r = \frac{10^{-3}}{10^{4/3}} \approx 4.64 \times 10^{-5} \text{ meters}$$

$$46. a) E_r = \begin{cases} \frac{\sigma_a}{\epsilon r} & a < r < b \\ 0 & r > b \end{cases} \rightarrow W = \int_a^b \frac{1}{2} \epsilon E_r^2 2\pi r dr$$

$$= \frac{\pi (\sigma_a)^2}{\epsilon} \int_a^b \frac{dr}{r}$$

$$= \frac{\pi (\sigma_a)^2}{\epsilon} \ln \frac{b}{a}$$

$$b) E_r = \begin{cases} \frac{\rho_a r}{2\epsilon} & 0 < r < a \\ \frac{\rho_a a^2}{2\epsilon r} & a < r < b \end{cases} \rightarrow W = \int_0^a \frac{1}{2} \epsilon \left(\frac{\rho_a r}{2\epsilon}\right)^2 2\pi r dr + \int_a^b \frac{1}{2} \epsilon \left(\frac{\rho_a a^2}{2\epsilon r}\right)^2 2\pi r dr$$

$$= \frac{\pi \rho_a^2 a^4}{4\epsilon} \left[ \frac{1}{4} + \ln \frac{b}{a} \right]$$

$$c) E_r = \begin{cases} \frac{\rho_a r}{2\epsilon} & 0 < r < a \\ \frac{\rho_a a^2}{2\epsilon r} + \frac{\rho_b (r^2 - a^2)}{2\epsilon r} & a < r < b \end{cases}$$

$$W = \int_0^a \frac{1}{2} \epsilon \left(\frac{\rho_a r}{2\epsilon}\right)^2 2\pi r dr + \int_a^b \frac{1}{2} \epsilon \left[ \frac{\rho_a a^2}{2\epsilon r} + \rho_b \frac{(r^2 - a^2)}{2\epsilon r} \right]^2 2\pi r dr$$

$$= \frac{\pi \rho_a^2 a^4}{16\epsilon} + \frac{\pi a^4}{4\epsilon} (\rho_a - \rho_b)^2 \ln \frac{b}{a} + \frac{\pi \rho_b^2}{16\epsilon} (b^4 - a^4) + \frac{\pi \rho_b}{2\epsilon} (\rho_a - \rho_b) a^2 (b - a)$$

POLARIZATION AND CONDUCTION

$$47. \quad a) \quad W = \frac{1}{2} Q V_-(r=0) - \frac{1}{2} Q [V_+(R) + V_-(R)]$$

$$V_-(r) = \begin{cases} \frac{-Q}{4\pi\epsilon_o r} & r \geq R \\ \frac{-Q}{4\pi\epsilon_o R} & r \leq R \end{cases} ; \quad V_+(r) = \frac{Q}{4\pi\epsilon_o r}$$

$$W = \frac{1}{2} Q \left[ \frac{-Q}{4\pi\epsilon_o R} \right] - \frac{1}{2} Q [0]$$

$$= \frac{-Q^2}{8\pi\epsilon_o R}$$

$$b) \quad V(r) = \begin{cases} \frac{3Q}{8\pi\epsilon_o R_1^3} (R_1^2 - \frac{r^2}{3}) - \frac{Q}{4\pi\epsilon_o R_2} & r < R_1 \\ \frac{Q}{4\pi\epsilon_o r} - \frac{Q}{4\pi\epsilon_o R_2} & R_1 < r < R_2 \end{cases}$$

$$\begin{aligned} W &= \frac{1}{2} \int_{r=0}^{R_1} \rho V r^2 \sin\theta dr d\theta d\phi + \frac{1}{2} \int \sigma_f(R_2) V(R_2) r^2 \sin\theta d\theta d\phi \\ &= \frac{1}{2} \int_{r=0}^{R_1} \frac{Q}{\frac{4}{3}\pi R_1^3} \left[ \frac{3Q}{8\pi\epsilon_o R_1^3} (R_1^2 - \frac{r^2}{3}) - \frac{Q}{4\pi\epsilon_o R_2} \right] 4\pi r^2 dr \\ &= \frac{Q^2}{8\pi\epsilon_o R_1} \left[ \frac{6}{5} - \frac{R_1}{R_2} \right] \end{aligned}$$

Check:

$$E_r = \begin{cases} \frac{Qr}{4\pi\epsilon_o R_1^3} & 0 < r < R_1 \\ \frac{Q}{4\pi\epsilon_o r^2} & R_1 < r < R_2 \\ 0 & r > R_2 \end{cases}$$

$$W = \frac{1}{2} \epsilon_o \int_0^{R_2} |E_r|^2 4\pi r^2 dr$$

# POLARIZATION AND CONDUCTION

$$= 2\pi\epsilon_o \left(\frac{Q}{4\pi\epsilon_o}\right)^2 \left\{ \int_0^{R_1} \frac{r^4}{R_1^6} dr + \int_{R_1}^{R_2} \frac{1}{r^2} dr \right\} = \frac{Q^2}{8\pi\epsilon_o R_1} \left[ \frac{6}{5} - \frac{R_1}{R_2} \right]$$

48. a)  $W_{init} = \frac{1}{2} C v_1^2(t=0) + \frac{1}{2} C v_2^2(t=0) = \frac{1}{2} C V_o^2$

b)  $\frac{v_1 - v_2}{R} = i = \frac{C dv_2}{dt} = - \frac{C dv_1}{dt}; \quad R = \frac{\ell}{\sigma A}$

$$v_2 + v_1 = V_o \rightarrow \frac{dv_1}{dt} + \frac{2v_1}{RC} = \frac{V_o}{RC}$$

$$v_1 = \frac{V_o}{2} [1 + e^{-t/\tau}]; \quad v_2 = \frac{V_o}{2} [1 - e^{-t/\tau}]; \quad \tau = \frac{RC}{2}$$

$$i = \frac{V_o}{R} e^{-t/\tau}$$

c)  $\lim_{t \rightarrow \infty} v_1 = v_2 = \frac{V_o}{2}, \quad i = 0$

$$W_{final} = \frac{1}{2} C v_1^2 + \frac{1}{2} C v_2^2 = \frac{1}{4} C V_o^2 = \frac{1}{2} W_{init}$$

d)  $W_{diss} = \int_0^\infty i^2 R dt$

$$= \frac{V_o^2}{R} \int_0^\infty e^{-2t/\tau} dt$$

$$= - \frac{V_o^2 \tau}{2R} e^{-2t/\tau} \Big|_0^\infty$$

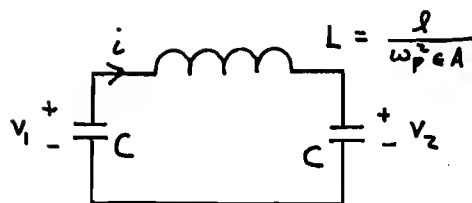
$$= \frac{1}{4} C V_o^2 = \frac{1}{2} W_{init}$$

e) If  $R = 0 \rightarrow \tau = 0$ . From (d) the dissipated energy is independent of  $R$ . The circuit problem becomes nonphysical as an infinite current flows for zero time.

f)  $\frac{di}{dt} = \omega_p^2 \epsilon A E \rightarrow \frac{C d^2 v_2}{dt^2} = - \frac{C d^2 v_1}{dt^2} = \frac{di}{dt} = \frac{\omega_p^2 \epsilon A}{\ell} (v_1 - v_2)$

# POLARIZATION AND CONDUCTION

$$v_1 + v_2 = V_o \rightarrow \frac{Cd^2 v_1}{dt^2} + \frac{2\omega_p^2 \epsilon A}{\ell} v_1 = \frac{\omega_p^2 \epsilon AV_o}{\ell}$$



$$\begin{aligned} \text{g) } \frac{Ld^2 i}{dt^2} + \frac{2i}{C} &= 0 \rightarrow i(t) = I_1 \sin \omega_o t + I_2 \cos \omega_o t, \quad \omega_o = \sqrt{\frac{2}{LC}} \\ &= \sqrt{\frac{2\omega_p^2 \epsilon A}{\ell C}} \end{aligned}$$

$$i(t=0) = 0 \rightarrow I_2 = 0$$

$$\rightarrow i(t) = \frac{V_o}{L\omega_o} \sin \omega_o t$$

$$L \left. \frac{di}{dt} \right|_{t=0} = V_o = LI_1 \omega_o$$

$$v_1(t) = -\frac{1}{C} \int i dt = -\frac{V_o}{2} [\cos \omega_o t + 1]$$

$$v_2(t) = \frac{1}{C} \int i dt = -\frac{V_o}{2} [\cos \omega_o t - 1]$$

$$\text{h) } W_1 = \frac{1}{2} C v_1^2 = \frac{1}{8} C V_o^2 [1 + 2 \cos \omega_o t + \cos^2 \omega_o t]$$

$$W_2 = \frac{1}{2} C v_2^2 = \frac{1}{8} C V_o^2 [1 - 2 \cos \omega_o t + \cos^2 \omega_o t]$$

$$W_L = \frac{1}{2} L i^2 = \frac{1}{2} \frac{V_o^2}{L \omega_o^2} [\sin^2 \omega_o t] = \frac{1}{4} C V_o^2 \sin^2 \omega_o t$$

$$\begin{aligned} \text{i) } W_{\text{total}} &= W_1 + W_2 + W_L = \frac{1}{4} C V_o^2 [1 + \sin^2 \omega_o t + \cos^2 \omega_o t] \\ &= \frac{1}{2} C V_o^2 \end{aligned}$$

Section 3.9

49. a)  $\vec{T} = \vec{r} \times \vec{F} \rightarrow T = qd\sin\theta E$

$$= d\vec{i}_z \times q\vec{E} = pE\sin\theta$$

b)  $dW = Td\theta \rightarrow W = pE \int_0^\theta \sin\theta d\theta$

$$= pE\sin\theta d\theta = -pE(\cos\theta - 1)$$

c)  $n = n_o e^{pE/kT} e^{pE\cos\theta/kT}$

$$N = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^R nr^2 \sin\theta dr d\theta d\phi$$

$$= n_o e^{pE/kT} \frac{2\pi R^3}{3} \int_{\theta=0}^{\pi} e^{pE\cos\theta/kT} \sin\theta d\theta$$

Let  $u = \frac{pE}{kT} \cos\theta$ ,  $du = -\frac{pE}{kT} \sin\theta d\theta$

$$N = -n_o e^{pE/kT} \frac{2\pi R^3}{3} \frac{kT}{pE} \int_{pE/kT}^{-pE/kT} e^u du$$

$$= -\frac{2\pi R^3 kT}{3pE} n_o e^{pE/kT} \left[ \underbrace{e^{-pE/kT} - e^{pE/kT}}_{-2\sinh \frac{pE}{kT}} \right]$$

$$n = \frac{3pEN}{4\pi R^3 kT \sinh \frac{pE}{kT}} e^{pE\cos\theta/kT}$$

d)  $dP_z = p n \cos\theta d\theta r^2 \sin\theta dr d\phi$

e)  $P_z = \frac{pN}{2\sinh \frac{pE}{kT}} \int_{\theta=0}^{\pi} \frac{pE}{kT} \cos\theta \sin\theta e^{pE\cos\theta/kT} d\theta$

$$= \frac{-pN}{2\sinh \frac{pE}{kT}} \int_{pE/kT}^{-pE/kT} \frac{kT}{pE} u e^u du$$



# POLARIZATION AND CONDUCTION

$$\begin{aligned}
 &= \frac{-pN}{2 \sinh \frac{pE}{kT}} \frac{kT}{pE} e^u (u - 1) \bigg|_{u = pE/kT}^{-pE/kT} \\
 &= \frac{-pN}{2 \sinh \frac{pE}{kT}} \frac{kT}{pE} [e^{-pE/kT} (-\frac{pE}{kT} - 1) - e^{pE/kT} (\frac{pE}{kT} - 1)] \\
 &= \frac{-pN}{\sinh \frac{pE}{kT}} \frac{kT}{pE} [-\frac{pE}{kT} \cosh \frac{pE}{kT} + \sinh \frac{pE}{kT}] \\
 &= -pN [\frac{kT}{pE} - \coth \frac{pE}{kT}]
 \end{aligned}$$

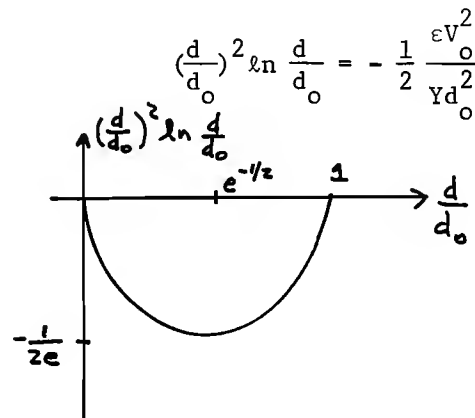
$$f) \frac{pE}{kT} \approx \frac{(1.6 \times 10^{-19})(10^{-9})10^6}{(1.38 \times 10^{-23})(300)} \approx .0386$$

$$\coth \frac{pE}{kT} \approx \frac{1}{pE/kT} [1 + \frac{1}{3}(\frac{pE}{kT})^2] \rightarrow P_z \approx \frac{pN}{3} (\frac{pE}{kT})$$

$$g) P_z = \alpha E \rightarrow \alpha = \frac{p^2 N}{3kT}$$

$$50. \text{ From (3.9.3) } f_x = \frac{1}{2} (\epsilon - \epsilon_o) \frac{V_o^2 D}{s} = \rho_m g h s D \rightarrow h = \frac{1}{2} \frac{(\epsilon - \epsilon_o) V_o^2}{\rho_m g s^2}$$

$$51. (a)-(c) \sigma_f = \frac{\epsilon V_o}{d} \rightarrow f_e = \frac{1}{2} \sigma_f E = \frac{1}{2} \frac{\epsilon V_o^2}{d^2} = -Y \ln \frac{d}{d_o}$$



The minimum value of  $\left(\frac{d}{d_o}\right)^2 \ln \frac{d}{d_o}$  is  $-\frac{1}{2e}$  at  $\frac{d}{d_o} = e^{-1/2}$ . This value is reached when

$$V_o^2 = \frac{d_o^2 Y}{\epsilon e} = \frac{Y d_o^2}{\epsilon}$$

# POLARIZATION AND CONDUCTION

52. a)  $E = E_1$  (in electret);  $E = E_2$  (in free space)

$$E_1 d + E_2 s = V_o \quad E_1 = (V_o - \frac{P_o s}{\epsilon_o}) \frac{1}{s + d}$$

→

$$\epsilon_o E_1 + P_o = \epsilon_o E_2 \quad E_2 = (V_o + \frac{P_o d}{\epsilon_o}) \frac{1}{s + d}$$

b)  $f_x = \frac{1}{2} \sigma_f E_2 A = \frac{1}{2} \epsilon_o E_2^2 A = \frac{1}{2} \frac{\epsilon_o A}{(s + d)^2} [V_o + \frac{P_o d}{\epsilon_o}]^2$

53. a)  $\nabla \cdot \vec{E} = \frac{dE_x}{dx} = \frac{\rho_o}{\epsilon} \rightarrow E_x = \frac{\rho_o x}{\epsilon} + C$

$$\int_0^s E_x dx = \frac{\rho_o s^2}{2\epsilon} + Cs = V_o \rightarrow C_1 = \frac{V_o}{s} - \frac{\rho_o s}{2\epsilon} \rightarrow E_x = \frac{\rho_o}{\epsilon} (x - \frac{s}{2}) + \frac{V_o}{s}$$

$$W = \int_0^s \frac{1}{2} \epsilon E_x^2 A dx = \frac{1}{2} \epsilon A \int_0^s \left[ \frac{\rho_o (x - \frac{s}{2})}{\epsilon} + \frac{V_o}{s} \right]^2 dx$$

$$= \frac{1}{2} \epsilon^2 A \left[ \frac{\rho_o (x - \frac{s}{2})}{\epsilon} + \frac{V_o}{s} \right]^3 \Big|_{x=0}^s$$

$$= \frac{\epsilon^2 A}{6\rho_o} \left\{ \left[ \frac{\rho_o s}{2\epsilon} + \frac{V_o}{s} \right]^3 - \left[ -\frac{\rho_o s}{2\epsilon} + \frac{V_o}{s} \right]^3 \right\}$$

$$= \frac{\epsilon A s}{6} \left[ \left( \frac{\rho_o s}{2\epsilon} \right)^2 + 3 \left( \frac{V_o}{s} \right)^2 \right]$$

b)  $\frac{1}{2} C V_o^2 = W \rightarrow C = \frac{2W}{V_o^2} = \frac{\epsilon A s}{3V_o^2} \left[ \left( \frac{\rho_o s}{2\epsilon} \right)^2 + 3 \left( \frac{V_o}{s} \right)^2 \right]$

$$\lim_{\rho_o=0} C = \frac{\epsilon A}{s}$$

c)  $\sigma_f(x=0) = \epsilon E_x(x=0); \sigma_f(x=s) = -\epsilon E_x(x=s)$

$$f_x(x=0) = \frac{1}{2} \sigma_f(x=0) E_x(x=0) A = \frac{1}{2} \epsilon E_x^2(x=0) = \frac{1}{2} \epsilon \left[ -\frac{\rho_o s}{2\epsilon} + \frac{V_o}{s} \right]^2 A$$

$$f_x(x=s) = \frac{1}{2} \sigma_f(x=s) E_x(x=s) A = -\frac{1}{2} \epsilon E_x^2(x=s) = -\frac{1}{2} \epsilon \left[ \frac{\rho_o s}{2\epsilon} + \frac{V_o}{s} \right]^2 A$$

POLARIZATION AND CONDUCTION

$$\begin{aligned}
 f_{vol} &= \int_0^s \rho_o A E_x(x) dx \\
 &= \int_0^s \epsilon A E_x \frac{dE_x}{dx} dx \\
 &= \frac{1}{2} \epsilon E_x^2 \Big|_{x=0}^s \\
 &= \frac{1}{2} \epsilon E_x^2(x=s) - \frac{1}{2} \epsilon E_x^2(x=0)
 \end{aligned}$$

$$d) \quad f_x(x=0) + f_x(x=s) + f_{vol} = 0$$

$$54. \quad a) \quad E_r(z=z_o) = \frac{V_o}{r \ln \frac{b}{a}}, \quad E_r(z=-\infty) = -\frac{P_o}{\epsilon_o}$$

$$F_z = P_r \frac{\partial E_z}{\partial r}; \quad \nabla \times \vec{E} = 0 \rightarrow \frac{\partial E_r}{\partial z} = \frac{\partial E_z}{\partial r}$$

$$\begin{aligned}
 F_z = P_o \frac{\partial E_r}{\partial z} \rightarrow f_z &= \int_{\phi=0}^{2\pi} \int_{r=a}^b \int_{z=-\infty}^{z_o} P_o \frac{\partial E_r}{\partial z} r dr d\phi dz \\
 &= \int_{r=a}^b P_o E_r \Big|_{z=-\infty}^{z_o} 2\pi r dr \\
 &= 2\pi P_o \int_{r=a}^b \left[ \frac{V_o}{r \ln \frac{b}{a}} + \frac{P_o}{\epsilon_o} \right] r dr \\
 &= 2\pi P_o \left[ \frac{V_o (b-a)}{\ln \frac{b}{a}} + \frac{P_o}{\epsilon_o} \frac{(b^2 - a^2)}{2} \right]
 \end{aligned}$$

$$b) \quad C(z) = \frac{2\pi}{\ln \frac{b}{a}} [\epsilon z + \epsilon_o (\ell - z)]$$

$$f_z = \frac{1}{2} V_o^2 \frac{dC}{dz} = \frac{\pi V_o^2}{\ln \frac{b}{a}} (\epsilon - \epsilon_o)$$

$$55. \quad C(x) = \frac{\epsilon_o (\ell - x)d}{s}; \quad f_x = \frac{1}{2} V_o^2 \frac{dC}{dx} = -\frac{1}{2} \frac{\epsilon_o d}{s} V_o^2$$

56. a)  $p = vi = v \frac{d}{dt} [C(\theta)v]$

$$= \frac{d}{dt} \left[ \frac{1}{2} C(\theta)v^2 \right] + \underbrace{\frac{1}{2} v^2 \frac{dC(\theta)}{d\theta} \frac{d\theta}{dt}}_T$$

$$T = \frac{1}{2} v^2 \frac{dC}{d\theta}, \quad W = \frac{1}{2} C(\theta)v^2$$

b)  $C(\theta) = \frac{\epsilon_o 2NR^2}{s} (\theta_o - \theta)$

c)  $T = \frac{1}{2} v^2 \frac{dC}{d\theta} = - \frac{NV_o^2 R^2 \epsilon_o}{s}$

d)  $T_m + T_e = 0 \rightarrow - \frac{NV_o^2 R^2 \epsilon_o}{s} - K(\theta - \theta_s) = 0 \rightarrow \theta - \theta_s = - \frac{NV_o^2 R^2 \epsilon_o}{Ks}$

At high frequency, the shaft position responds to the time average of  $v^2(t)$  (rms value).

e)  $C(\theta) = \frac{1}{2} [C_{\max} + C_{\min}] + \frac{1}{2} [C_{\max} - C_{\min}] \cos 2\theta$

$$T = \frac{1}{2} v_o^2 \frac{dC}{d\theta} = - \frac{1}{2} v_o^2 (C_{\max} - C_{\min}) \sin 2\theta$$

$$= - \frac{1}{2} v_o^2 (C_{\max} - C_{\min}) \cos^2 \omega t \sin 2\theta$$

f)  $T = - \frac{1}{2} v_o^2 (C_{\max} - C_{\min}) \left[ \frac{1}{2} \sin 2\theta + \frac{1}{4} [\sin(2(\omega t + \theta)) - \sin(2(\omega t - \theta))] \right]$

T has a time average value if  $\omega = \pm \omega_m$  ( $\theta = \omega_m t + \delta$ ).

g)  $T_o = - \frac{1}{8} v_o^2 (C_{\max} - C_{\min}) \sin 2\delta$

$$\rightarrow \delta = - \frac{1}{2} \sin^{-1} \left[ \frac{8T_o}{v_o^2 (C_{\max} - C_{\min})} \right]$$

h)  $|T_{\max}| = \frac{1}{8} v_o^2 (C_{\max} - C_{\min})$  at  $\delta = \pm 45^\circ, \pm 135^\circ$

Section 3.10

$$57. \quad a) \quad i = \sigma_f U w = C \frac{dv}{dt} \rightarrow v(t) = \frac{\sigma_f U w t}{C} ; \quad C = 4\pi\epsilon_o R$$

b) Force on the charged belt is

$$F = \sigma_f \frac{v(t)}{\ell} w \ell = \sigma_f w v(t)$$

$$P_{\text{mech}} = FU = \sigma_f w v(t) U = \frac{(\sigma_f U w)^2 t}{C}$$

$$58. \quad a) \quad U \frac{\partial \rho_f}{\partial z} + \frac{\sigma}{\epsilon} \rho_f = 0 \rightarrow \rho_f = \rho_o e^{-\sigma z / \epsilon U}$$

$$\frac{dE}{dz} = \frac{\rho_f}{\epsilon} \rightarrow E = -\frac{\rho_o U}{\sigma} e^{-\sigma z / \epsilon U} + C = -\frac{\rho_f U}{\sigma} + C$$

$$i = J w t = \underbrace{[\rho_f U + \sigma E] w t}_{\sigma C} = \frac{V_o}{R_L} \rightarrow C = \frac{V_o}{\sigma R_L w t}$$

$$b) \quad \int_0^{\ell} E dz = -V_o = \frac{\rho_o \epsilon U^2}{\sigma^2} e^{-\sigma z / \epsilon U} \Big|_0^{\ell} + C \ell$$

$$= \frac{\rho_o \epsilon U^2}{\sigma^2} [e^{-\sigma \ell / \epsilon U} - 1] + \frac{V_o \ell}{\sigma R_L w t}$$

$$V_o = \frac{\frac{\rho_o \epsilon U^2}{\sigma^2} [1 - e^{-\sigma \ell / \epsilon U}]}{1 + \frac{\ell}{\sigma w t R_L}}$$

$$59. \quad a) \quad -nC_i v_2 - C \frac{dv_1}{dt} - \frac{v_1}{R} - \frac{(v_1 - v_2)}{R_L} = 0$$

$$-nC_i v_1 - C \frac{dv_2}{dt} - \frac{v_2}{R} + \frac{(v_1 - v_2)}{R_L} = 0$$

$$v_1 = V_1 e^{st}, \quad v_2 = V_2 e^{st}$$

$$V_1 [Cs + \frac{1}{R} + \frac{1}{R_L}] + V_2 [nC_i - \frac{1}{R_L}] = 0$$

## POLARIZATION AND CONDUCTION

$$v_1 [nC_i - \frac{1}{R_L}] + v_2 [Cs + \frac{1}{R} + \frac{1}{R_L}] = 0$$

$$[Cs + \frac{1}{R} + \frac{1}{R_L}]^2 = [nC_i - \frac{1}{R_L}]^2$$

$$s = -\frac{1}{C} [\frac{1}{R} + \frac{1}{R_L} \pm \frac{1}{R_L} \mp nC_i]$$

$$s_1 = -\frac{1}{C} [\frac{1}{R} + \frac{2}{R_L} - nC_i], \quad s_2 = -\frac{1}{C} [\frac{1}{R} + nC_i]$$

$$s_1 > 0 \rightarrow nC_i > \frac{1}{R} + \frac{2}{R_L} \quad (\text{self-excited})$$

$$b) \quad n = 10, \quad C_i = 2 \text{ pf}, \quad C = 10 \text{ pf}, \quad R_L = R$$

$$R_{\min} = \frac{3}{nC_i} = 1.5 \times 10^{11} \Omega$$

$$c) \quad nC_i v_1 + C \frac{dv_2}{dt} + \frac{v_2}{R} = 0; \quad v_1 = V_1 e^{st}$$

$$nC_i v_2 + C \frac{dv_3}{dt} + \frac{v_3}{R} = 0; \quad v_2 = V_2 e^{st}$$

$$nC_i v_3 + C \frac{dv_1}{dt} + \frac{v_1}{R} = 0; \quad v_3 = V_3 e^{st}$$

$$\begin{bmatrix} nC_i & Cs + \frac{1}{R} & 0 \\ 0 & nC_i & Cs + \frac{1}{R} \\ Cs + \frac{1}{R} & 0 & nC_i \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

$$(nC_i)^3 + (Cs + \frac{1}{R})^3 = 0 \rightarrow s = -\frac{1}{RC} - e^{j2\pi r/3} \frac{nC_i}{C}, \quad r = 1, 2, 3$$

# POLARIZATION AND CONDUCTION

For self-excitation

$$\frac{nC_i}{2} > \frac{1}{R} \quad \text{with} \quad \omega_o = \frac{\sqrt{3}}{2} \frac{nC_i}{C}$$

$$d) \quad nC_i V_k + (Cs + \frac{1}{R})V_{k+1} = 0 ; \quad V_k = A\lambda^k$$

$$[Cs + \frac{1}{R}]\lambda + nC_i = 0 \rightarrow \lambda = - \frac{nC_i}{Cs + \frac{1}{R}}$$

$$\lambda^{N+1} = \lambda \rightarrow \lambda^N = 1 \rightarrow - \frac{nC_i}{Cs + \frac{1}{R}} = e^{j2\pi r/N}, \quad r = 1, 2, \dots, N$$

$$s = - \frac{nC_i}{C} e^{-j2\pi r/N} - \frac{1}{RC}$$

For self-excitation

$$\text{Re}(s) > 0 \rightarrow \cos 2\pi r/N > - \frac{1}{nRC_i}, \quad \omega_o = |\text{Im}(s)| = \left| \frac{nC_i}{C} \sin \frac{2\pi r}{N} \right|$$

## CHAPTER 4

### ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

#### Section 4.1

$$1. \quad a) \quad \vec{E} = -\frac{V_o}{ab} [y\vec{i}_x + x\vec{i}_y]; \quad \sigma_f(y=0) = \epsilon_o E_y(y=0) = -\frac{V_o x}{ab}$$

$$\sigma_f(x=0) = \epsilon_o E_x(x=0) = -\frac{V_o y}{ab}$$

$$q_T(x=0) = D \int_0^{y_o} \sigma_f(x=0) dy = -\frac{V_o D y_o^2}{2ab}$$

$$q_T(y=0) = D \int_0^{x_o} \sigma_f(y=0) dx = -\frac{V_o D x_o^2}{2ab}$$

$$q_T = q_T(x=0) + q_T(y=0) = -\frac{V_o D}{2ab} (x_o^2 + y_o^2)$$

$$C = \left| \frac{q_T}{V_o} \right| = \frac{D}{2ab} (x_o^2 + y_o^2)$$

$$b) \quad \frac{1}{2} mv^2 + qV = \text{constant} = qV_o \rightarrow v = \sqrt{\frac{2q}{m} (V_o - V)}$$

$$= \sqrt{\frac{2qV_o}{m} \left(1 - \frac{xy}{ab}\right)}$$

$$c) \quad v_{\text{impact}} = \sqrt{\frac{2qV_o}{m}}$$

$$2. \quad a) \quad V = \begin{cases} [A_1 \sin ay + A_2 \cos ay] e^{-ax} & x > 0 \\ [B_1 \sin ay + B_2 \cos ay] e^{+ax} & x < 0 \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} ae^{-ax} \{ [A_1 \sin ay + A_2 \cos ay] \vec{i}_x - [A_1 \cos ay - A_2 \sin ay] \vec{i}_y \} & x > 0 \\ -ae^{+ax} \{ [B_1 \sin ay + B_2 \cos ay] \vec{i}_x + [B_1 \cos ay - B_2 \sin ay] \vec{i}_y \} & x < 0 \end{cases}$$



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Boundary conditions:

$$E_y(x=0_+) = E_y(x=0_-) \rightarrow A_1 = B_1, A_2 = B_2$$

$$\epsilon[E_x(x=0_+) - E_x(x=0_-)] = \sigma_f = \sigma_o \cos \alpha \rightarrow A_1 = B_1 = 0, A_2 = B_2 = \frac{\sigma_o}{2a\epsilon}$$

$$V = \begin{cases} \frac{\sigma_o}{2a\epsilon} \cos \alpha e^{-ax} & x > 0 \\ \frac{\sigma_o}{2a\epsilon} \cos \alpha e^{+ax} & x < 0 \end{cases}, \quad \vec{E} = -\nabla V = \begin{cases} \frac{\sigma_o e^{-ax}}{2\epsilon} [\cos \alpha \vec{i}_x + \sin \alpha \vec{i}_y] & x > 0 \\ -\frac{\sigma_o e^{+ax}}{2\epsilon} [\cos \alpha \vec{i}_x - \sin \alpha \vec{i}_y] & x < 0 \end{cases}$$

$$b) \frac{dy}{dx} = \frac{E_y}{E_x} = \pm \tan \alpha \rightarrow e^{\pm ax} \sin \alpha y = \text{constant}$$

$$3. a) V = \begin{cases} V_o \cos \alpha e^{a(x + \frac{d}{2})} & x < -d/2 \\ -V_o \cos \alpha \frac{\sinh ax}{\sinh \frac{ad}{2}} & |x| < d/2 \\ -V_o \cos \alpha e^{-a(x - \frac{d}{2})} & x > d/2 \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} -V_o a e^{a(x + \frac{d}{2})} [\cos \alpha \vec{i}_x - \sin \alpha \vec{i}_y] & x < -d/2 \\ \frac{V_o a}{\sinh \frac{ad}{2}} [\cosh ax \cos \alpha \vec{i}_x - \sinh ax \sin \alpha \vec{i}_y] & |x| < d/2 \\ -V_o a e^{-a(x - \frac{d}{2})} [\cos \alpha \vec{i}_x + \sin \alpha \vec{i}_y] & x > d/2 \end{cases}$$

$$b) \sigma_f(x = -\frac{d}{2}) = \epsilon[E_x(x = -\frac{d}{2}^+) - E_x(x = -\frac{d}{2}^-)]$$

$$= \epsilon V_o a \cos \alpha [\coth \frac{ad}{2} + 1]$$

$$\sigma_f(x = \frac{d}{2}) = \epsilon[E_x(x = \frac{d}{2}^+) - E_x(x = \frac{d}{2}^-)]$$

$$= -\epsilon V_o a \cos \alpha [1 + \coth \frac{ad}{2}]$$

$$4. a) V(x, y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sin \frac{n\pi y}{d} [A_n \cosh \frac{n\pi x}{d} + B_n \sinh \frac{n\pi x}{d}]$$

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$$V(x=0) = V_1 = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sin \frac{n\pi y}{d} A_n \rightarrow A_n = \frac{4V_1}{n\pi}$$

$$V(x=\ell) = 0 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sin \frac{n\pi y}{d} [A_n \cosh \frac{n\pi \ell}{d} + B_n \sinh \frac{n\pi \ell}{d}]$$

$$B_n = -A_n \coth \frac{n\pi \ell}{d}$$

$$V(x,y) = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n} [\cosh \frac{n\pi x}{d} - \coth \frac{n\pi \ell}{d} \sinh \frac{n\pi x}{d}]$$

$$= \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n \sinh \frac{n\pi \ell}{d}} \sinh \frac{n\pi(\ell - x)}{d}$$

$$\vec{E} = -\nabla V = -\frac{4V_1}{d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{\sinh \frac{n\pi \ell}{d}} [\cos \frac{n\pi y}{d} \sinh \frac{n\pi(\ell-x)}{d} \vec{i}_y - \sin \frac{n\pi y}{d} \cosh \frac{n\pi(\ell-x)}{d} \vec{i}_x]$$

$$b) \quad V = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n \sinh \frac{n\pi \ell}{d}} \sinh \frac{n\pi(\ell-x)}{d} + \frac{4V_2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi x}{\ell} \sinh \frac{n\pi(d-y)}{\ell}}{n \sinh \frac{n\pi d}{\ell}}$$

$$c) \quad V = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d} \sinh \frac{n\pi(\ell-x)}{d}}{n \sinh \frac{n\pi \ell}{d}} + \frac{4V_2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi x}{\ell} \sinh \frac{n\pi(d-y)}{\ell}}{n \sinh \frac{n\pi d}{\ell}}$$

$$+ \frac{4V_3}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d} \sinh \frac{n\pi x}{d}}{n \sinh \frac{n\pi \ell}{d}} + \frac{4V_4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi x}{\ell} \sinh \frac{n\pi y}{\ell}}{n \sinh \frac{n\pi d}{\ell}}$$

$$5. \quad a) \quad c = \sqrt{a^2 + b^2}$$

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$$V(x,y,z) = \begin{cases} \frac{V_0 \sin ax \cos bz \sinh c(y-d)}{\sinh c(s-d)} & s \leq y \leq d \\ \frac{V_0 \sin ax \cos bz \sinh cy}{\sinh cs} & 0 \leq y \leq s \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} \frac{-V_0}{\sinh c(s-d)} [a \cos ax \cos bz \sinh c(y-d) \vec{i}_x - b \sin ax \sin bz \sinh c(y-d) \vec{i}_z + c \sin ax \cos bz \cosh c(y-d) \vec{i}_y] & s < y < d \\ \frac{-V_0}{\sinh cs} [a \cos ax \cos bz \sinh cy \vec{i}_x - b \sin ax \sin bz \sinh cy \vec{i}_z + c \sin ax \cos bz \cosh cy \vec{i}_y] & 0 < y < s \end{cases}$$

$$b) \quad \sigma_f(y=0) = \epsilon E_y(y=0) = \frac{-\epsilon V_0 c}{\sinh cs} \sin ax \cos bz$$

$$\sigma_f(y=d) = -\epsilon E_y(y=d) = \frac{\epsilon V_0 c}{\sinh c(s-d)} \sin ax \cos bz$$

$$\begin{aligned} \sigma_f(y=s) &= \epsilon [E_y(y=s_+) - E_y(y=s_-)] \\ &= -\epsilon V_0 c \sin ax \cos bz [\coth c(s-d) - \coth cs] \end{aligned}$$

$$6. \quad \sigma_f = \frac{4\sigma_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi y}{d}$$

$$a) \quad v = \begin{cases} \sum_{n \text{ odd}} V_n \sin \frac{n\pi y}{d} e^{-n\pi x/d} & x \geq 0 \\ \sum_{n \text{ odd}} V_n \sin \frac{n\pi y}{d} e^{+n\pi x/d} & x \leq 0 \end{cases}$$

$$\vec{E} = -\nabla v = \begin{cases} - \sum_{n \text{ odd}} V_n \frac{n\pi}{d} [\cos \frac{n\pi y}{d} \vec{i}_y - \sin \frac{n\pi y}{d} \vec{i}_x] e^{-n\pi x/d} & x > 0 \\ - \sum_{n \text{ odd}} V_n \frac{n\pi}{d} [\cos \frac{n\pi y}{d} \vec{i}_y + \sin \frac{n\pi y}{d} \vec{i}_x] e^{+n\pi x/d} & x < 0 \end{cases}$$

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$$\sigma_f = \epsilon [E_x(x=0_+) - E_x(x=0_-)] \rightarrow \frac{4\sigma_o}{n\pi} = 2\epsilon V_n \frac{n\pi}{d}$$

$$V_n = \frac{2\sigma_o d}{\epsilon(n\pi)^2} \quad n = 1, 3, 5, \dots$$

$$b) \quad \sigma_f(y=0) = \epsilon E_y(y=0) = \begin{cases} -\sum_{n \text{ odd}} \frac{2\sigma_o}{(n\pi)} e^{-n\pi x/d} & x > 0 \\ -\sum_{n \text{ odd}} \frac{2\sigma_o}{(n\pi)} e^{+n\pi x/d} & x < 0 \end{cases}$$

$$\sigma_f(y=d) = -\epsilon E_y(y=d) = \sigma_f(y=0)$$

$$c) \quad q_T(y=0) = q_T(y=d) = 2 \int_0^{\infty} \sigma_f(y=0) dx$$

$$= -4\sigma_o \sum_{n \text{ odd}} \frac{1}{n\pi} \int_0^{\infty} e^{-(n\pi x/d)} dx$$

$$= 4\sigma_o \sum_{n \text{ odd}} \frac{d}{(n\pi)^2} e^{-n\pi x/d} \Big|_{x=0}^{\infty}$$

$$= -4\sigma_o d \sum_{n \text{ odd}} \underbrace{\left(\frac{1}{n\pi}\right)^2}_{1/8} = -\frac{\sigma_o d}{2}$$

$$7. \quad a) \quad \nabla^2 V_p = \frac{d^2 V_p}{dx^2} = -\frac{\rho_o}{\epsilon_o} \sin ax \rightarrow V_p = \frac{\rho_o}{\epsilon_o a^2} \sin ax$$

$$V_p(y=0) \neq 0 \text{ (Boundary condition not satisfied)}$$

$$b) \quad \nabla^2 V = \frac{-\rho_o}{\epsilon_o} \sin ax \rightarrow V = \begin{cases} A \sin ax e^{-a(y-d)} & y \geq d \\ [B \sinh ay + C \cosh ay] \sin ax + \frac{\rho_o}{\epsilon_o a^2} \sin ax & 0 \leq y \leq d \end{cases}$$

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$$\bar{E} = -\nabla V = \begin{cases} -Aa[\cos ax \bar{i}_x - \sin ax \bar{i}_y]e^{-a(y-d)} & y \geq d \\ -\frac{\rho_o}{\epsilon_o a} \cos ax \bar{i}_x - a\{\cos ax[B\sinh ay + C\cosh ay]\bar{i}_x \\ + \sin ax[B\cosh ay + C\sinh ay]\bar{i}_y\} & 0 < y \leq d \end{cases}$$

## Boundary Conditions

$$V(y=0) = 0 \rightarrow \frac{\rho_o}{\epsilon_o a^2} + C = 0$$

$$V(y=d_-) = V(y=d_+) \rightarrow A = \frac{\rho_o}{\epsilon_o a^2} + B \sinh ad + C \cosh ad$$

$$\sigma_f(y=d) = \epsilon_o [E_y(y=d_+) - E_y(y=d_-)] = 0 \rightarrow B \cosh ad + C \sinh ad = -A$$

$$C = \frac{-\rho_o}{\epsilon_o a^2}, \quad A = \frac{\rho_o}{\epsilon_o a^2} \frac{(1 - e^{-ad})^2}{2}, \quad B = \frac{\rho_o}{\epsilon_o a^2} [1 - e^{-ad}]$$

$$c) \quad \sigma_f(y=0) = \epsilon_o E_y(y=0) = -\epsilon_o a \sin ax B = -\frac{\rho_o}{a} (1 - e^{-ad}) \sin ax$$

$$d) \quad F_y = \frac{1}{2} \sigma_f(y=0) E_y(y=0) = \frac{1}{2} \frac{\sigma_f(y=0)^2}{\epsilon_o} = \frac{1}{2} \frac{\rho_o^2}{\epsilon_o a^2} (1 - e^{-ad})^2 \sin^2 ax$$

$$\begin{aligned} f_y &= \int_0^{2\pi/a} F_y dx \\ &= \frac{1}{2} \frac{\rho_o^2}{\epsilon_o a^2} (1 - e^{-ad})^2 \int_0^{2\pi/a} \sin^2 ax dx \\ &= \frac{\pi}{2} \frac{\rho_o^2}{\epsilon_o a^3} (1 - e^{-ad})^2 \end{aligned}$$

$$\bar{F}_v = \rho_f \bar{E} = \rho_f [E_x \bar{i}_x + E_y \bar{i}_y]$$

$$\begin{aligned} f_{vx} &= \int_{x=0}^{2\pi/a} \int_{y=0}^d \rho_f E_x dx dy = -\rho_o a \int_{x=0}^{2\pi/a} \int_{y=0}^d [B \sinh ay + C \cosh ay + \frac{\rho_o}{\epsilon_o a^2}] \sin ax \cos ax dx dy \\ &= 0 \end{aligned}$$

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when integrated over  $x \left[ \int_0^{2\pi/a} \sin ax \cos ax \, dx = 0 \right]$ .

$$\begin{aligned}
 f_{vy} &= \int_{x=0}^{2\pi/a} \int_{y=0}^d \rho_f E_y \, dx dy \\
 &= -\rho_o a \int_{x=0}^{2\pi/a} \int_{y=0}^d \sin^2 ax \{B \cosh ay + C \sinh ay\} \, dx dy \\
 &= -\frac{\rho_o \pi}{a} [B \sinh ay + C \cosh ay] \Big|_{y=0}^d \\
 &= -\frac{\rho_o \pi}{a} [B \sinh ad + C(\cosh ad - 1)] \\
 &= -\frac{\rho_o \pi}{2 \epsilon_o a^3} (1 - e^{-ad})^2 = -f_y
 \end{aligned}$$

The total force on the system is zero ( $f_y + f_{vy} = 0$ ).

e)  $\nabla \cdot \bar{D} = 0 = \nabla \cdot [\epsilon_o \bar{E} + \bar{P}]$

$$\nabla \cdot \bar{P} = \frac{\partial P_y}{\partial y} = 0 \rightarrow \nabla \cdot \bar{E} = 0 \rightarrow \nabla^2 V = 0$$

$$V = \begin{cases} A \sin ax \, e^{-a(y-d)} & y \geq d \\ [B \sinh ay + C \cosh ay] \sin ax & 0 \leq y \leq d \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -Aa [\cos ax \bar{i}_x - \sin ax \bar{i}_y] e^{-a(y-d)} & y \geq d \\ -a \{ \cos ax [B \sinh ay + C \cosh ay] \bar{i}_x \\ + \sin ax [B \cosh ay + C \sinh ay] \bar{i}_y \} & 0 < y < d \end{cases}$$

Boundary Conditions:

$$V(y=0) = 0 \rightarrow C = 0$$

$$V(y=d_+) = V(y=d_-) \rightarrow A = B \sinh ad + C \cosh ad$$

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$$D_y(y=d_+) = D_y(y=d_-) \rightarrow \epsilon_0 E_y(y=d_+) = \epsilon_0 E_y(y=d_-) + P_o \sin ax$$

$$A = \frac{P_o}{\epsilon_0 a} e^{-ad} \sinh ad, \quad B = \frac{P_o}{\epsilon_0 a} e^{-ad}, \quad C = 0$$

$$\sigma_f(y=0) = \epsilon_0 E_y(y=0) = -a\epsilon_0 B \sin ax = -P_o e^{-ad} \sin ax$$

8. a) i)  $z^2 = (x + jy)^2 = (x^2 - y^2) + 2jxy \rightarrow u = x^2 - y^2, \quad v = 2xy$

ii)  $\sin z = \sin(x + jy) = \sin x \cos jy + \cos x \sin jy$   
 $= \sin x \cosh y + j \cos x \sinh y \rightarrow u = \sin x \cosh y, v = \cos x \sinh y$

iii)  $\cos z = \cos(x + jy) = \cos x \cos jy - \sin x \sin jy$   
 $= \cos x \cosh y - j \sin x \sinh y \rightarrow u = \cos x \cosh y, v = -\sin x \sinh y$

iv)  $e^z = e^{(x+jy)} = e^x [\cos y + j \sin y] \rightarrow u = e^x \cos y, v = e^x \sin y$

v)  $\ln z = \ln(x + jy) = \ln \sqrt{x^2 + y^2} + j \tan^{-1} \frac{y}{x} \rightarrow u = \ln \sqrt{x^2 + y^2}, v = \tan^{-1} \frac{y}{x}$

b)  $\frac{dw}{dz} = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} = \frac{1}{j} \left[ \frac{\partial u}{\partial y} + j \frac{\partial v}{\partial y} \right]$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

c)  $\frac{\partial}{\partial y} \left[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right] \rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$

$$\frac{\partial}{\partial x} \left[ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \right] \rightarrow -\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right] \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial}{\partial y} \left[ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \right] \rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}$$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

d)  $\nabla u = \frac{\partial u}{\partial x} \bar{i}_x + \frac{\partial u}{\partial y} \bar{i}_y$  is perpendicular to lines of constant  $u$

$\nabla v = \frac{\partial v}{\partial x} \bar{i}_x + \frac{\partial v}{\partial y} \bar{i}_y$  is perpendicular to lines of constant  $v$

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} = 0$$

$\nabla u \perp \nabla v \rightarrow$  lines of constant  $u$  and  $v$  are perpendicular.

## Section 4.3

9. a) 
$$V(\phi) = \begin{cases} A_1 \phi + B_1 & 0 < \phi < \alpha \\ A_2 \phi + B_2 & \alpha < \phi < \pi \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -\frac{A_1}{r} \bar{i}_\phi & 0 < \phi < \alpha \\ -\frac{A_2}{r} \bar{i}_\phi & \alpha < \phi < \pi \end{cases}$$

Boundary conditions:

$$V(\phi=0) = V_o = B_1$$

$$V(\phi=\pi) = 0 = A_2 \pi + B_2$$

$$V(\phi=\alpha_-) = V(\phi=\alpha_+) \rightarrow A_1 \alpha + B_1 = A_2 \alpha + B_2$$

At  $t = 0$

$$\sigma_f(\phi=\alpha) = \epsilon_2 E_\phi(\phi=\alpha_+) - \epsilon_1 E_\phi(\phi=\alpha_-) = 0 \rightarrow \epsilon_2 A_2 = \epsilon_1 A_1$$

$$A_1 = \frac{\epsilon_2}{\epsilon_1} A_2 = \frac{-\epsilon_2 V_o}{\epsilon_2 \alpha + \epsilon_1 (\pi - \alpha)}; \quad B_2 = -A_2 \pi, \quad B_1 = V_o$$



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At  $t = \infty$

$$J_{\phi}(\phi=\alpha_+) = J_{\phi}(\phi=\alpha_-) \rightarrow \sigma_2 E_{\phi}(\phi=\alpha_+) = \sigma_1 E_{\phi}(\phi=\alpha_-) \rightarrow \sigma_2 A_2 = \sigma_1 A_1$$

$$A_1 = \frac{\sigma_2}{\sigma_1} A_2 = \frac{-\sigma_2 V_o}{\sigma_2 \alpha + \sigma_1 (\pi - \alpha)}, \quad B_2 = -A_2 \pi, \quad B_1 = V_o$$

## Transient Interval

$$J_{\phi}(\phi=\alpha_+) - J_{\phi}(\phi=\alpha_-) + \frac{\partial}{\partial t} \sigma_f(\phi=\alpha) = 0$$

$$\sigma_2 E_{\phi}(\phi=\alpha_+) - \sigma_1 E_{\phi}(\phi=\alpha_-) + \frac{\partial}{\partial t} [\epsilon_2 E_{\phi}(\phi=\alpha_+) - \epsilon_1 E_{\phi}(\phi=\alpha_-)] = 0$$

$$\frac{\partial A_1}{\partial t} + \frac{A_1}{\tau} = - \frac{\sigma_2 V_o}{\epsilon_2 \alpha + \epsilon_1 (\pi - \alpha)}; \quad \tau = \frac{\epsilon_2 \alpha + \epsilon_1 (\pi - \alpha)}{\sigma_2 \alpha + \sigma_1 (\pi - \alpha)}$$

$$A_1 = - \frac{\epsilon_2 V_o e^{-t/\tau}}{\epsilon_2 \alpha + \epsilon_1 (\pi - \alpha)} - \frac{\sigma_2 V_o (1 - e^{-t/\tau})}{\sigma_2 \alpha + \sigma_1 (\pi - \alpha)}$$

$$A_2 = \frac{A_1 \alpha + V_o}{\alpha - \pi}, \quad B_1 = V_o, \quad B_2 = -A_2 \alpha$$

b)  $\sigma_f(\phi=\alpha) = \epsilon_2 E_{\phi}(\phi=\alpha_+) - \epsilon_1 E_{\phi}(\phi=\alpha_-)$

$$\begin{aligned} &= \frac{-\epsilon_2 A_2 + \epsilon_1 A_1}{r} = \frac{(\epsilon_2 - \sigma_2 \tau) V_o (1 - e^{-t/\tau})}{(\pi - \alpha) r} \\ &= \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2) V_o (1 - e^{-t/\tau})}{[\sigma_2 \alpha + \sigma_1 (\pi - \alpha)] r} \end{aligned}$$

c)  $\sigma_f(\phi=0) = \epsilon_1 E_{\phi}(\phi=0) = - \frac{\epsilon_1 A_1}{r}$

$$q(\phi=0) = \int_{r=a}^b \sigma_f \ell dr = -\epsilon_1 A_1 \ell \ln \frac{b}{a}$$

$$C = \frac{q(\phi=0, t=0)}{V_o} = - \frac{\epsilon_1 \ell \ln \frac{b}{a} A_1(t=0)}{V_o} = \frac{\epsilon_1 \epsilon_2 \ell \ln \frac{b}{a}}{\epsilon_2 \alpha + \epsilon_1 (\pi - \alpha)}$$

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$$\begin{aligned}
 R &= \frac{V_o}{I(t=\infty)}, \quad I(t=\infty) = \int_{r=0}^a J_r(t=\infty) \ell dr \\
 &= \int_{r=0}^a - \frac{\sigma_1 A_1(t=\infty) \ell dr}{r} \\
 &= - \sigma_1 \ell A_1(t=\infty) \ell n \frac{b}{a} = \frac{\sigma_1 \sigma_2 \ell \ell n \frac{b}{a} V_o}{\sigma_2 \alpha + \sigma_1 (\pi - \alpha)}
 \end{aligned}$$

$$R = \frac{V_o}{I(t=\infty)} = \frac{\sigma_2 \alpha + \sigma_1 (\pi - \alpha)}{\sigma_1 \sigma_2 \ell \ell n \frac{b}{a}}$$

10. a) 
$$V(r, \phi) = \begin{cases} V_o (r/a)^n \sin n\phi & 0 < r < a \\ V_o (r/a)^{-n} \sin n\phi & r > a \end{cases}$$

$$\begin{aligned}
 \vec{E} &= -\nabla V = - \frac{\partial V}{\partial r} \vec{i}_r - \frac{1}{r} \frac{\partial V}{\partial \phi} \vec{i}_\phi \\
 &= \begin{cases} - \frac{V_o n}{a} \left\{ \left(\frac{r}{a}\right)^{n-1} [\sin n\phi \vec{i}_r + \cos n\phi \vec{i}_\phi] \right\} & 0 < r < a \\ \frac{V_o n}{a} \left\{ \left(\frac{r}{a}\right)^{-n-1} [\sin n\phi \vec{i}_r - \cos n\phi \vec{i}_\phi] \right\} & r > a \end{cases}
 \end{aligned}$$

b) 
$$V(r=a, \phi) = \sum_{n=1}^{\infty} V_n \sin n\phi \rightarrow V_n = \frac{2V_o}{n\pi} \quad n \text{ odd}$$

c) 
$$V(r, \phi) = \begin{cases} \sum_{n \text{ odd}} \frac{2V_o}{n\pi} \left(\frac{r}{a}\right)^n \sin n\phi & 0 < r < a \\ \sum_{n \text{ odd}} \frac{2V_o}{n\pi} \left(\frac{r}{a}\right)^{-n} \sin n\phi & r > a \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} \sum_{n \text{ odd}} - \frac{2V_o n}{a} \left\{ \left(\frac{r}{a}\right)^{n-1} [\sin n\phi \vec{i}_r + \cos n\phi \vec{i}_\phi] \right\} & 0 < r < a \\ \sum_{n \text{ odd}} \frac{2V_o n}{a} \left\{ \left(\frac{r}{a}\right)^{-n-1} [\sin n\phi \vec{i}_r - \cos n\phi \vec{i}_\phi] \right\} & r > a \end{cases}$$

ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

11.

$$V(r, \phi) = \begin{cases} A_1 r \cos \phi & 0 \leq r \leq a \\ (A_2 r + \frac{B_2}{r}) \cos \phi & a \leq r \leq b \\ (A_3 r + \frac{B_3}{r}) \cos \phi & r \geq b \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -A_1 [\bar{i}_r \cos \phi - \bar{i}_\phi \sin \phi] & 0 < r < a \\ -(A_2 - \frac{B_2}{r^2}) \cos \phi \bar{i}_r + (A_2 + \frac{B_2}{r^2}) \sin \phi \bar{i}_\phi & a < r < b \\ -(A_3 - \frac{B_3}{r^2}) \cos \phi \bar{i}_r + (A_3 + \frac{B_3}{r^2}) \sin \phi \bar{i}_\phi & r > b \end{cases}$$

Boundary conditions:

$$V(r=a_-) = V(r=a_+) \rightarrow A_1 a = A_2 a + \frac{B_2}{a}$$

$$V(r=b_-) = V(r=b_+) \rightarrow A_2 b + \frac{B_2}{b} = A_3 b + \frac{B_3}{b}$$

$$\epsilon E_r(r=a_+) = \epsilon_0 E_r(r=a_-) \rightarrow \epsilon (A_2 - \frac{B_2}{a^2}) = \epsilon_0 A_1$$

$$\epsilon_0 E_r(r=b_+) = \epsilon E_r(r=b_-) \rightarrow \epsilon (A_2 - \frac{B_2}{b^2}) = \epsilon_0 (A_3 - \frac{B_3}{b^2})$$

$$\bar{E}(r \rightarrow \infty) = E_0 \bar{i}_x = E_0 [\bar{i}_r \cos \phi - \bar{i}_\phi \sin \phi] \rightarrow A_3 = -E_0$$

$$A_1 = \frac{-4E_0}{2(1 + \frac{a^2}{b^2}) + (\frac{\epsilon}{\epsilon_0} + \frac{\epsilon_0}{\epsilon})(1 - \frac{a^2}{b^2})}, \quad A_2 = \frac{A_1}{2} (1 + \frac{\epsilon_0}{\epsilon}), \quad B_2 = \frac{a^2 A_1}{2} (1 - \frac{\epsilon_0}{\epsilon})$$

$$A_3 = -E_0, \quad B_3 = -E_0 - \frac{\epsilon}{\epsilon_0} \frac{A_1}{2} [1 + \frac{\epsilon_0}{\epsilon} - \frac{a^2}{b^2}]$$

12.

$$V(r, \phi) = \begin{cases} A r \cos \phi & 0 \leq r \leq a \\ (B r + \frac{C}{r}) \cos \phi & r \geq a \end{cases}$$

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

$$\bar{E} = -\nabla V = \begin{cases} -A(\cos\phi \bar{i}_r - \sin\phi \bar{i}_\phi) & 0 \leq r < a \\ -(B - \frac{C}{2r})\cos\phi \bar{i}_r + (B + \frac{C}{2r})\sin\phi \bar{i}_\phi & r > a \end{cases}$$

Boundary conditions:

$$V(r=a_-) = V(r=a_+) \rightarrow Aa = Ba + \frac{C}{a}$$

$$\bar{E}(r \rightarrow \infty) = E_o(\bar{i}_r \cos\phi - \bar{i}_\phi \sin\phi) \rightarrow B = -E_o$$

$$\epsilon_o E_r(r=a_+) + P_1 \cos\phi = \epsilon_o E_r(r=a_-) + P_2 \cos\phi$$

$$V(r, \phi) = \begin{cases} [\frac{P_2 - P_1}{2\epsilon_o} - E_o]r \cos\phi & 0 \leq r \leq a \\ [-E_o r + \frac{(P_2 - P_1)a^2}{2\epsilon_o r}] \cos\phi & r > a \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -[\frac{P_2 - P_1}{2\epsilon_o} - E_o][\cos\phi \bar{i}_r - \sin\phi \bar{i}_\phi] & 0 \leq r < a \\ [E_o + \frac{a^2(P_2 - P_1)}{2\epsilon_o r^2}]\cos\phi \bar{i}_r - [E_o - \frac{a^2(P_2 - P_1)}{2\epsilon_o r^2}]\sin\phi \bar{i}_\phi & r > a \end{cases}$$

$$13. \ a) \ \bar{E} = [E_o(1 + \frac{a^2}{r^2})\cos\phi + \frac{\lambda(t)}{2\pi\epsilon r}] \bar{i}_r - E_o(1 - \frac{a^2}{r^2})\sin\phi \bar{i}_\phi$$

$$\bar{E} = 0 \rightarrow E_o(1 + \frac{a^2}{r^2})\cos\phi + \frac{\lambda(t)}{2\pi\epsilon r} = 0$$

$$E_o(1 - \frac{a^2}{r^2})\sin\phi = 0 \rightarrow \phi = 0, \pi$$

$r = a$

At  $r = a$

$$2E_o \cos\phi + \frac{\lambda(t)}{2\pi\epsilon a} \rightarrow \cos\phi_c = \frac{-\lambda(t)}{4\pi\epsilon E_o a}$$

At  $\phi = 0, \pi$

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

$$\pm E_o \left(1 + \frac{a^2}{r^2}\right) + \frac{\lambda(t)}{2\pi\epsilon r} = 0 \rightarrow r^2 + a^2 \pm \frac{\lambda(t)r}{2\pi\epsilon E_o} = 0$$

$$r = \mp \frac{\lambda(t)}{4\pi\epsilon E_o} \pm \sqrt{\left(\frac{\lambda}{4\pi\epsilon E_o}\right)^2 - a^2}$$

$$\frac{1}{r} \frac{\partial \Sigma}{\partial \phi} = E_r = E_o \left(1 + \frac{a^2}{r^2}\right) \cos \phi + \frac{\lambda(t)}{2\pi\epsilon r}$$

$$-\frac{\partial \Sigma}{\partial r} = E_\phi = -E_o \left(1 - \frac{a^2}{r^2}\right) \sin \phi$$

$$\rightarrow \frac{\Sigma}{E_o a} = \frac{\left(\frac{r}{a}\right)^2 + 1}{\left(\frac{r}{a}\right)} \sin \phi + \frac{\lambda}{2\pi\epsilon E_o a} \phi = \text{constant}$$

b)  $E_r(r=a) < 0 \rightarrow 2E_o \cos \phi + \frac{\lambda(t)}{2\pi\epsilon a} < 0 \rightarrow \cos \phi < \frac{-\lambda(t)}{4\pi\epsilon a E_o}$

c)  $-1 \leq \cos \phi \leq 1 \rightarrow \lambda_{\max} = 4\pi\epsilon a E_o$

d)  $dI = -q_o n_o \mu E_r(r=a) a d\phi$

$$I = -q_o n_o \mu a \int_{\phi_c}^{2\pi - \phi_c} \left[ 2E_o \cos \phi + \frac{\lambda}{2\pi\epsilon a} \right] d\phi; \quad \phi_c = \cos^{-1} \frac{\lambda}{4\pi\epsilon a E_o}$$

$$= -q_o n_o \mu a \left[ 2E_o \sin \phi + \frac{\lambda \phi}{2\pi\epsilon a} \right] \Bigg|_{\phi_c}^{2\pi - \phi_c}$$

$$= -q_o n_o \mu a \left[ -4E_o \sin \phi_c + \frac{\lambda}{\pi\epsilon a} (\pi - \phi_c) \right]$$

e)  $\lim_{\substack{r \rightarrow \infty \\ \phi \rightarrow \pi}} y^* = r \sin \phi = \frac{\Sigma}{E_o} - \frac{\lambda \pi}{2\pi\epsilon E_o}$

$$\frac{\Sigma}{E_o} = 2a \sin \phi_c + \frac{\lambda \phi_c}{2\pi\epsilon E_o}$$

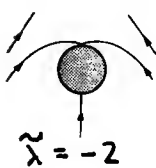
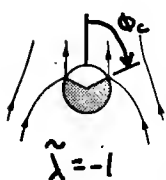
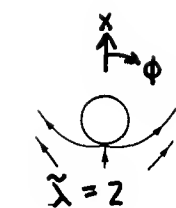
$$y^* = 2a \sin \phi_c + \frac{\lambda(\phi_c - \pi)}{2\pi\epsilon E_o}$$

$$I = q_o n_o \mu E_o (2y^*)$$

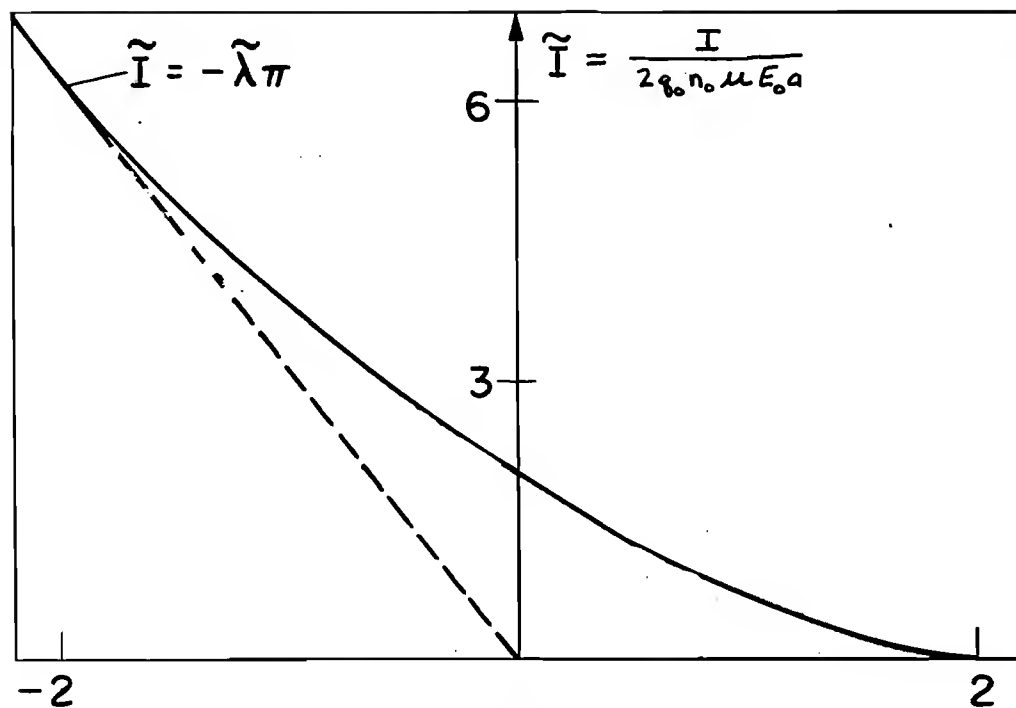
$$= 2q_o n_o \mu E_o a \left[ 2 \sin \phi_c + \frac{\lambda(\phi_c - \pi)}{2\pi\epsilon E_o a} \right]$$

(agrees with d)

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS



(a)



$$\tilde{\chi} = \frac{\lambda}{2\pi\epsilon_0\epsilon_a}$$

(d), (e)

(d)

14. a)  $\sigma_f(t) = \sigma_f(t=0)e^{-t/\tau}; \quad \tau = \frac{\epsilon_1 + \epsilon_2}{\sigma_1 + \sigma_2}$

$$= \frac{2(\sigma_2 \epsilon_1 - \sigma_1 \epsilon_2)}{\sigma_1 + \sigma_2} E_0 \cos \phi e^{-t/\tau}$$

b)  $V = \text{Re} \hat{V}(r, \phi) e^{j\omega t}$

$$\hat{V}(r, \phi) = \begin{cases} \hat{A} \cos \phi & 0 \leq r \leq a \\ [\hat{B}r + \frac{\hat{C}}{r}] \cos \phi & r \geq a \end{cases}$$

$$\hat{E} = -\nabla \hat{V} = \begin{cases} -\hat{A}[\cos \phi \bar{i}_r - \sin \phi \bar{i}_\phi] & 0 \leq r < a \\ -[\hat{B} - \frac{\hat{C}}{r^2}] \cos \phi \bar{i}_r + [\hat{B} + \frac{\hat{C}}{r^2}] \sin \phi \bar{i}_\phi & r > a \end{cases}$$

$$\hat{E}(r \rightarrow \infty) = E_0(\bar{i}_r \cos \phi - \bar{i}_\phi \sin \phi) \rightarrow \hat{B} = -E_0$$

$$\hat{V}(r=a_+) = \hat{V}(r=a_-) \rightarrow \hat{A}a = \hat{B}a + \frac{\hat{C}}{a}$$

$$\hat{J}_r(r=a_+) - \hat{J}_r(r=a_-) + j\omega \hat{\sigma}_f = 0 \rightarrow [\sigma_1 + j\omega \epsilon_1] \hat{E}_r(r=a_+) = [\sigma_2 + j\omega \epsilon_2] \hat{E}_r(r=a_-)$$

$$\hat{C} = a^2 E_0 \frac{[\sigma_2 - \sigma_1 + j\omega(\epsilon_2 - \epsilon_1)]}{\sigma_2 + \sigma_1 + j\omega(\epsilon_2 + \epsilon_1)}, \quad \hat{B} = -E_0, \quad \hat{A} = \frac{2E_0(\sigma_1 + j\omega \epsilon_1)}{\sigma_1 + \sigma_2 + j\omega(\epsilon_1 + \epsilon_2)}$$

$$\hat{\sigma}_f = \epsilon_1 \hat{E}_r(r=a_+) - \epsilon_2 \hat{E}_r(r=a_-)$$

$$= [-\epsilon_1(\hat{B} - \frac{\hat{C}}{a^2}) + \epsilon_2 \hat{A}] \cos \phi$$

$$= \frac{2E_0[\epsilon_1 \sigma_2 - \epsilon_2 \sigma_1] \cos \phi}{\sigma_1 + \sigma_2 + j\omega(\epsilon_1 + \epsilon_2)}$$

15. a)  $V(r, z) = \begin{cases} A_1 z + B_1 z \ln r + C_1 \ln r + D_1 & a \leq r \leq b \\ A_2 z + B_2 z \ln r + C_2 \ln r + D_2 & b \leq r \leq c \end{cases}$

Boundary conditions:

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

$$V(r, z=0) = 0 \quad 0 \leq r \leq c$$

$$V(r=a, z) = 0 \quad 0 \leq z \leq \ell$$

$$V(r, z=\ell) = V_o \quad b \leq r \leq c$$

$$J_r(r=b, z) = 0 \quad 0 \leq z \leq \ell$$

$$\rightarrow V(r, z) = \begin{cases} \frac{V_o z}{\ell \ln \frac{b}{a}} \ln \frac{r}{a} & a \leq r \leq b \\ \frac{V_o z}{\ell} & b \leq r \leq c \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} \frac{-V_o}{\ell \ln \frac{b}{a}} \left[ \ln \frac{r}{a} \vec{i}_z + \frac{z}{r} \vec{i}_r \right] & a < r < b \\ \frac{-V_o}{\ell} \vec{i}_z & b \leq r \leq c \end{cases}$$

$$b) \quad \sigma_f(r=a) = \epsilon_o E_r(r=a) = \frac{-\epsilon_o V_o}{\ell \ln \frac{b}{a}} \frac{z}{a}$$

$$\sigma_f(r=b) = \epsilon E_r(r=b_+) - \epsilon_o E_r(r=b_-) = \frac{\epsilon_o V_o}{\ell \ln \frac{b}{a}} \frac{z}{b}$$

$$\sigma_f(z=0) = \begin{cases} \epsilon_o E_z(z=0) = \frac{-\epsilon_o V_o}{\ell \ln \frac{b}{a}} \ln \frac{r}{a} & a < r < b \\ \epsilon E_z(z=0) = \frac{-\epsilon V_o}{\ell} & b \leq r \leq c \end{cases}$$

$$c) \quad \frac{dr}{dz} = \frac{E_r}{E_z} = \frac{z}{r \ln \frac{r}{a}} \rightarrow z^2 = r^2 \left[ \ln \frac{r}{a} - \frac{1}{2} \right] + \text{constant}$$

$$16. \quad a) \quad V(r, z) = \begin{cases} A_1 z + B_1 z \ln r + C_1 \ln r + D_1 & 0 \leq r \leq a \\ A_2 z + B_2 z \ln r + C_2 \ln r + D_2 & a \leq r \leq b \end{cases}$$

Boundary conditions:

$$V(z=0) = 0 \rightarrow C_1 = D_1 = 0, \quad C_2 = D_2 = 0$$

$$V(r=b) = 0 \rightarrow A_2 + B_2 \ln b = 0$$

$$V(z=\ell, r < a) = V_o \rightarrow A_1 \ell = V_o, \quad B_1 = 0$$

$$V(r=a_+) = V(r=a_-) \rightarrow A_2 + B_2 \ln a = A_1$$



# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

$$A_1 = \frac{V_0}{\ell}, \quad B_1 = C_1 = D_1 = C_2 = D_2 = 0, \quad B_2 = \frac{V_0}{\ell \ln \frac{a}{b}}, \quad A_2 = -\frac{V_0 \ln b}{\ell \ln \frac{a}{b}}$$

$$V(r, z) = \begin{cases} \frac{V_0 z}{\ell} \\ \frac{V_0 z \ln \frac{b}{a}}{\ell \ln \frac{b}{a}} \end{cases}; \quad \bar{E} = -\nabla V = \begin{cases} -\frac{V_0}{\ell} \bar{i}_z \\ -\frac{V_0}{\ell \ln \frac{b}{a}} \left[ \ln \frac{b}{r} \bar{i}_z - \frac{z}{r} \bar{i}_r \right] \end{cases} \quad \begin{matrix} 0 \leq r \leq a \\ a < r < b \end{matrix}$$

$$b) \quad \sigma_f(r=a) = \epsilon_0 E_r(r=a_+) - \epsilon_0 E_r(r=a_-)$$

$$= \frac{\epsilon_0 V_0}{\ell \ln \frac{b}{a}} \frac{z}{a}$$

$$q_T(r=a) = 2\pi a \int_0^{\ell} \sigma_f dz$$

$$= \frac{\pi \epsilon_0 V_0 \ell}{\ln \frac{b}{a}}$$

$$c) \quad \frac{dr}{dz} = \frac{E_r}{E_z} = \frac{-z}{r \ln \frac{b}{r}} \rightarrow z^2 = r^2 \left[ \ln \frac{r}{b} - \frac{1}{2} \right] + \text{constant}$$

## Section 4.4

$$17. \quad a) \quad \bar{E} = E_0 \left\{ \left(1 + \frac{2R^3}{r^3}\right) \cos\theta \bar{i}_r - \left(1 - \frac{R^3}{r^3}\right) \sin\theta \bar{i}_\theta \right\}$$

$$\sigma_f = \epsilon_0 E_r(r=R) = 3\epsilon_0 E_0 \cos\theta$$

$$\begin{aligned} q_t &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sigma_f R^2 \sin\theta d\theta d\phi \\ &= 3\epsilon_0 E_0 R^2 2\pi \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta d\theta \\ &= -3E_0 R^2 2\pi \epsilon_0 \left. \frac{\cos 2\theta}{4} \right|_{\theta=0}^{\pi/2} \\ &= 3\pi \epsilon_0 E_0 R^2 \end{aligned}$$

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

If the hemisphere were not there  $q_t = \pi R^2 \epsilon_0 E_0$  so that the hemisphere has three times more charge.

$$b) F_z = \int_{\theta=0}^{\pi/2} \frac{1}{2} \epsilon E_r^2(r=R) \cos\theta 2\pi R^2 \sin\theta d\theta$$

$$= 9\pi \epsilon R^2 E_0^2 \int_{\theta=0}^{\pi/2} \cos^3\theta \sin\theta d\theta$$

$$= 9\pi \epsilon R^2 E_0^2 \left. \frac{-\cos^4\theta}{4} \right|_0^{\pi/2}$$

$$= \frac{9}{4} \pi \epsilon R^2 E_0^2 \geq \frac{2}{3} \pi R^3 \rho_m g \rightarrow E_0 \geq \sqrt{\frac{8 \rho_m g R}{27 \epsilon}}$$

$$18. a) V(r, \theta) = \begin{cases} A(t)r \cos\theta & r \leq R \\ [B(t)r + \frac{C(t)}{r^2}] \cos\theta & r \geq R \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} -A(t)[\vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta] & r < R \\ -(B(t) - \frac{2C(t)}{r^3}) \cos\theta \vec{i}_r + (B(t) + \frac{C(t)}{r^3}) \sin\theta \vec{i}_\theta & r > R \end{cases}$$

Boundary conditions:

$$\vec{E}(r \rightarrow \infty) = E_0 (\vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta) \rightarrow B(t) = -E_0$$

$$V(r=R_+) = V(r=R_-) \rightarrow A(t)R = BR + \frac{C(t)}{R^2}$$

$$\sigma_1 E_r(r=R_+) - \sigma_2 E_r(r=R_-) + \frac{\partial}{\partial t} [\epsilon_1 E_r(r=R_+) - \epsilon_2 E_r(r=R_-)] = 0$$

Initial conditions:

$$\sigma_f(t=0) = \epsilon_1 E_r(r=R_+) - \epsilon_2 E_r(r=R_-) = 0$$

$$A(t=0) = -\frac{3\epsilon_1 E_0}{2\epsilon_1 + \epsilon_2}, B = -E_0, C(t=0) = \frac{(\epsilon_2 - \epsilon_1)R^3}{2\epsilon_1 + \epsilon_2} E_0$$

$$b) \frac{\partial C}{\partial t} + \frac{C}{\tau} = \frac{(\sigma_2 - \sigma_1)E_0 R^3}{2\epsilon_1 + \epsilon_2}; \tau = \frac{2\epsilon_1 + \epsilon_2}{2\sigma_1 + \sigma_2}$$

$$C(t) = E_o R^3 \left\{ \frac{\sigma_2 - \sigma_1}{2\sigma_1 + \sigma_2} (1 - e^{-t/\tau}) + \frac{(\epsilon_2 - \epsilon_1)}{2\epsilon_1 + \epsilon_2} e^{-t/\tau} \right\}$$

$$A(t) = -E_o + \frac{C(t)}{R^3}$$

$$\begin{aligned} \text{c) } \sigma_f &= \epsilon_1 E_r(r=R_+) - \epsilon_2 E_r(r=R_-) \\ &= \left\{ \epsilon_1 \left[ -B + \frac{2C}{R^3} \right] + \epsilon_2 A \right\} \cos\theta \\ &= \left\{ (\epsilon_1 - \epsilon_2) E_o + (2\epsilon_1 + \epsilon_2) \frac{C}{R^3} \right\} \cos\theta \\ &= \frac{3(\epsilon_1 \sigma_2 - \epsilon_2 \sigma_1)}{2\sigma_1 + \sigma_2} E_o \cos\theta (1 - e^{-t/\tau}) \end{aligned}$$

d) Charge conservation boundary condition at  $r = R$  becomes

$$[\sigma_1 + j\omega\epsilon_1] \hat{E}_r(r=R_+) = [\sigma_2 + j\omega\epsilon_2] \hat{E}_r(r=R_-); \quad \bar{E} = \text{Re} \hat{E} e^{j\omega t}$$

The solution is the same as letting

$$\epsilon_1 \rightarrow \sigma_1 + j\omega\epsilon_1, \quad \epsilon_2 \rightarrow \sigma_2 + j\omega\epsilon_2$$

in the initial condition solutions of (a)

$$\hat{A} = \frac{-3[\sigma_1 + j\omega\epsilon_1]}{2\sigma_1 + \sigma_2 + j\omega(2\epsilon_1 + \epsilon_2)}, \quad \hat{B} = -E_o, \quad \hat{C} = \frac{[(\sigma_2 - \sigma_1) + j\omega(\epsilon_2 - \epsilon_1)] R^3 E_o}{2\sigma_1 + \sigma_2 + j\omega(2\epsilon_1 + \epsilon_2)}$$

$$19. \quad V(r, \theta) = \begin{cases} Ar^2(3\cos^2\theta - 1) & r \leq R \\ \frac{B}{r^3} (3\cos^2\theta - 1) & r \geq R \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -2Ar[(3\cos^2\theta - 1)\bar{i}_r - 3\cos\theta\sin\theta\bar{i}_\theta] & r < R \\ \frac{3B}{r^4} [(3\cos^2\theta - 1)\bar{i}_r + 2\sin\theta\cos\theta\bar{i}_\theta] & r > R \end{cases}$$

Boundary conditions:

$$V(r=R_+) = V(r=R_-) \rightarrow AR^2 = \frac{B}{R^3}$$

$$\epsilon_1 E_r(r=R_+) - \epsilon_2 E_r(r=R_-) = \sigma_0 (3\cos^2\theta - 1) \rightarrow \frac{3B}{R^4} \epsilon_1 + 2AR\epsilon_2 = \sigma_0$$

$$B = \frac{\sigma_0 R^4}{[3\epsilon_1 + 2\epsilon_2]}, \quad A = \frac{\sigma_0}{R[3\epsilon_1 + 2\epsilon_2]}$$

$$20. \quad V(r, \theta) = \begin{cases} \text{Arcos}\theta & r \leq R \\ (Br + \frac{C}{r^2})\cos\theta & r \geq R \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} -A(\vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta) & r < R \\ -(B - \frac{2C}{r^3})\cos\theta \vec{i}_r + (B + \frac{C}{r^3})\sin\theta \vec{i}_\theta & r > R \end{cases}$$

Boundary conditions:

$$V(r=R_+) = V(r=R_-) \rightarrow AR = BR + \frac{C}{R^2}$$

$$\vec{E}(r \rightarrow \infty) = E_0 (\vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta) \rightarrow B = -E_0$$

$$\epsilon_0 E_r(r=R_+) + P_1 \cos\theta = \epsilon_0 E_r(r=R_-) + P_2 \cos\theta$$

$$A = -E_0 + \frac{(P_2 - P_1)}{3\epsilon_0}, \quad B = -E_0, \quad C = \frac{(P_2 - P_1)R^3}{3\epsilon_0}$$

$$21. \quad V(r, \theta) = \begin{cases} (Ar + \frac{B}{r^2})\cos\theta & r \leq R \\ \frac{C}{r^2} \cos\theta & r \geq R \end{cases}$$

$$\vec{E} = -\nabla V = \begin{cases} -(A - \frac{2B}{r^3})\cos\theta \vec{i}_r + (A + \frac{B}{r^3})\sin\theta \vec{i}_\theta & r < R \\ \frac{C}{r^3} [2\cos\theta \vec{i}_r + \sin\theta \vec{i}_\theta] & r > R \end{cases}$$

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

Boundary conditions:

$$\lim_{r \rightarrow 0} V(r, \theta) = \frac{p}{4\pi\epsilon_2} \frac{\cos\theta}{r^2} \rightarrow B = \frac{p}{4\pi\epsilon_2}$$

$$V(r=R_+) = V(r=R_-) \rightarrow AR + \frac{B}{R^2} = \frac{C}{R^2}$$

$$\epsilon_1 E_r(r=R_+) = \epsilon_2 E_r(r=R_-) \rightarrow -\epsilon_2 \left( A - \frac{2B}{R^3} \right) = \frac{2C}{R^3} \epsilon_1$$

$$A = \frac{2p(\epsilon_2 - \epsilon_1)}{4\pi\epsilon_2 R^3 (\epsilon_2 + 2\epsilon_1)}, \quad B = \frac{p}{4\pi\epsilon_2}, \quad C = \frac{3\epsilon_2}{\epsilon_2 + 2\epsilon_1} \frac{p}{4\pi\epsilon_2} = \frac{3p}{4\pi(\epsilon_2 + 2\epsilon_1)}$$

## Section 4.5

```

22.  DIMENSION V(4,4,20), P(4,4,20)
      READ (5,30) V1, V2, V3, V4
      WRITE (6,9)
      WRITE (6,8) V1, V2, V3, V4
      WRITE (6,9)
      DO 10 K=1,10
      DO 10 J=1,4
      DO 10 I=1,4
      V(I,J,K)=0. ; P(I,J,K)=0.
      V(I,1,K)=V1 ; V(4,J,K)=V2 ; V(I,4,K)=V3 ; V(1,J,K)=V4
10 CONTINUE
      DO 3 K=1,10
      DO 3 J=2,3
      DO 3 I=2,3
      V(I,J,K+1)=0.25*(V(I+1,J,K)+V(I-1,J,K)+V(I,J+1,K)+V(I,J-1,K))-2.
      P(I,J,K)=V(I,J,K) ; V(I,J,K)=V(I,J,K+1)
3 CONTINUE
      WRITE (6,7) (K,K=1,10)
      WRITE (6,9)
      WRITE (6,6) (P(2,2,K),K=1,10)
      WRITE (6,4)
      WRITE (6,6) (P(3,2,K),K=1,10)
      WRITE (6,4)
      WRITE (6,6) (P(2,3,K),K=1,10)
      WRITE (6,4)
      WRITE (6,6) (P(3,3,K),K=1,10)
      WRITE (6,9) ; WRITE (6,9) ; WRITE (6,9)
4 FORMAT (' ')
6 FORMAT ('+',10(F11.4,2X))
7 FORMAT ('+',6X,9(I2,11X),I2)
8 FORMAT ('OV1 = ',F7.2,4X,'V2 = ',F7.2,4X,'V3 = ',F7.2,4X,'V4 = ',
&F7.2)
9 FORMAT ('O')

```

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

```
30 FORMAT (4F10.2)
STOP
END
```

V1 = 0.00    V2 = 0.00    V3 = 0.00    V4 = 0.00

K	1	2	3	4	5
V(2,2)	0.0000	-2.0000	-3.2500	-3.8125	-3.9531
V(3,2)	0.0000	-2.5000	-3.6250	-3.9063	-3.9766
V(2,3)	0.0000	-2.5000	-3.6250	-3.9063	-3.9766
V(3,3)	0.0000	-3.2500	-3.8125	-3.9531	-3.9883
	6	7	8	9	10
	-3.9883	-3.9971	-3.9993	-3.9998	-4.0000
	-3.9941	-3.9985	-3.9996	-3.9999	-4.0000
	-3.9941	-3.9985	-3.9996	-3.9999	-4.0000
	-3.9971	-3.9993	-3.9998	-4.0000	-4.0000

```
23.  DIMENSION V(4,4,20), P(4,4,20)
      1 DO 5 M=1,2
        READ (5,30) V1, V2, V3, V4
        WRITE (6,9)
        WRITE (6,8) V1, V2, V3, V4
        WRITE (6,9)
        DO 10 K=1,10
          DO 10 J=1,4
            DO 10 I=1,4
              V(I,J,K)=0. ; P(I,J,K)=0.
              V(I,1,K)=V1 ; V(4,J,K)=V2 ; V(I,4,K)=V3 ; V(1,J,K)=V4
            10 CONTINUE
            DO 3 K=1,10
              DO 3 J=2,3
                DO 3 I=2,3
                  V(I,J,K+1)=0.25*(V(I+1,J,K)+V(I-1,J,K)+V(I,J+1,K)+V(I,J-1,K))
                  P(I,J,K)=V(I,J,K) ; V(I,J,K)=V(I,J,K+1)
                3 CONTINUE
                WRITE (6,7) (K,K=1,10)
                WRITE (6,9)
                WRITE (6,6) (P(2,2,K),K=1,10)
                WRITE (6,4)
                WRITE (6,6) (P(3,2,K),K=1,10)
                WRITE (6,4)
                WRITE (6,6) (P(2,3,K),K=1,10)
                WRITE (6,4)
                WRITE (6,6) (P(3,3,K),K=1,10)
              5 CONTINUE
              WRITE (6,9) ; WRITE (6,9) ; WRITE (6,9)
            4 FORMAT (' ')
            6 FORMAT ('+',10(F11.4,2X))
            7 FORMAT ('+',6X,9(I2,11X),I2)
```

# ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

```

8 FORMAT ('OV1 = ',F7.2,4X,'V2 = ',F7.2,4X,'V3 = ',F7.2,4X,'V4 = ',
&F7.2)
9 FORMAT ('0')
30 FORMAT (4F10.2)
STOP
END

```

V1 = 1.00    V2 = -2.00    V3 = 3.00    V4 = -4.00

K	1	2	3	4	5
V(2,2)	0.0000	-0.7500	-0.9688	-0.9922	-0.9980
V(3,2)	0.0000	-0.4375	-0.4844	-0.4961	-0.4990
V(2,3)	0.0000	-0.4375	-0.4844	-0.4961	-0.4990
V(3,3)	0.0000	0.0313	0.0078	0.0020	0.0005
	6	7	8	9	10
	-0.9995	-0.9999	-1.0000	-1.0000	-1.0000
	-0.4998	-0.4999	-0.5000	-0.5000	-0.5000
	-0.4998	-0.4999	-0.5000	-0.5000	-0.5000
	0.0001	0.0000	0.0000	0.0000	0.0000

V1 = 1.00    V2 = -2.00    V3 = -3.00    V4 = 4.00

K	1	2	3	4	5
V(2,2)	0.0000	1.2500	1.4063	1.2891	1.2598
V(3,2)	0.0000	0.0625	-0.1719	-0.2305	-0.2451
V(2,3)	0.0000	0.5625	0.3281	0.2695	0.2549
V(3,3)	0.0000	-1.0938	-1.2109	-1.2402	-1.2476
	6	7	8	9	10
	1.2524	1.2506	1.2502	1.2500	1.2500
	-0.2488	-0.2497	-0.2499	-0.2500	-0.2500
	0.2512	0.2503	0.2501	0.2500	0.2500
	-1.2494	-1.2498	-1.2500	-1.2500	-1.2500

CHAPTER 5  
THE MAGNETIC FIELD

Section 5.1

1. a)  $v_x = v_{x0} \cos \omega_o t + v_{y0} \sin \omega_o t$  ;  $\omega_o = qB_o/m$

$$v_y = v_{y0} \cos \omega_o t - v_{x0} \sin \omega_o t$$

$$v_z = v_{z0}$$

$$x = \int v_x dt = x_o + \frac{v_{y0}}{\omega_o} + \frac{1}{\omega_o} [-v_{y0} \cos \omega_o t + v_{x0} \sin \omega_o t]$$

$$y = \int v_y dt = y_o - \frac{v_{x0}}{\omega_o} + \frac{1}{\omega_o} [v_{y0} \sin \omega_o t + v_{x0} \cos \omega_o t]$$

$$z = \int v_z dt = v_{z0} t + z_o$$

b)  $[x - (x_o + \frac{v_{y0}}{\omega_o})]^2 + [y - (y_o - \frac{v_{x0}}{\omega_o})]^2 = \frac{v_{x0}^2 + v_{y0}^2}{\omega_o^2}$

Equation of circle with radius  $r = [(v_{x0}^2 + v_{y0}^2)/\omega_o^2]^{1/2}$  with center at  $(x_o + \frac{v_{y0}}{\omega_o}, y_o - \frac{v_{x0}}{\omega_o})$ .

c)  $\frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m [v_{x0}^2 + v_{y0}^2 + v_{z0}^2]$

2. a)  $m \frac{dv_x}{dt} = -e(-\frac{V_o}{s} + v_y B_o)$

$$m \frac{dv_y}{dt} = ev_x B_o \rightarrow v_x = \frac{m}{eB_o} \frac{dv_y}{dt}$$

$$\frac{d^2 v_y}{dt^2} + \omega_o^2 v_y = \frac{\omega_o^2 V_o}{B_o s} ; \quad \omega_o = \frac{eB_o}{m}$$



# THE MAGNETIC FIELD

$$v_y = A_1 \sin \omega_o t + A_2 \cos \omega_o t + \frac{V_o}{B_o s}$$

$$v_y(t=0) = 0 = A_2 + \frac{V_o}{B_o s}$$

$$v_x(t=0) = 0 \rightarrow \left. \frac{dv}{dt} \right|_{t=0} = 0 \rightarrow A_1 = 0$$

$$v_y = \frac{V_o}{B_o s} (1 - \cos \omega_o t), \quad v_x = \frac{1}{\omega_o} \frac{dv_y}{dt} = \frac{V_o}{B_o s} \sin \omega_o t$$

$$x = \int v_x dt = \frac{V_o}{B_o s \omega_o} (1 - \cos \omega_o t), \quad y = \int v_y dt = \frac{V_o}{B_o s} \left( t - \frac{\sin \omega_o t}{\omega_o} \right)$$

$$b) \quad x_{\max} = \frac{2V_o}{B_o s \omega_o} < s \rightarrow B_o^2 > \frac{2mV_o}{es^2} \quad \text{for cut-off}$$

$$c) \quad \bar{i}_r = \cos \phi \bar{i}_x + \sin \phi \bar{i}_y, \quad \bar{i}_\phi = -\sin \phi \bar{i}_x + \cos \phi \bar{i}_y$$

$$\frac{d\bar{i}_r}{dt} = [-\sin \phi \bar{i}_x + \cos \phi \bar{i}_y] \frac{d\phi}{dt} = \bar{i}_\phi \frac{v_\phi}{r}$$

$$\frac{d\bar{i}_\phi}{dt} = [-\cos \phi \bar{i}_x - \sin \phi \bar{i}_y] \frac{d\phi}{dt} = -\bar{i}_r \frac{v_\phi}{r}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \bar{i}_r \frac{dv_r}{dt} + v_r \frac{d\bar{i}_r}{dt} + \bar{i}_\phi \frac{dv_\phi}{dt} + v_\phi \frac{d\bar{i}_\phi}{dt}$$

$$= \bar{i}_r \left[ \frac{dv_r}{dt} - \frac{v_r^2}{r} \right] + \bar{i}_\phi \left[ \frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r} \right]$$

$$d) \quad m \frac{d\bar{v}}{dt} = -e[\bar{E} + \bar{v} \times \bar{B}]; \quad \bar{E} = \frac{-V_o}{r \ln \frac{b}{a}} \bar{i}_r$$

$$\left[ \frac{dv_r}{dt} - \frac{v_r^2}{r} \right] = \frac{-eE}{m} - \frac{ev_\phi B_o}{m} = \frac{eV_o}{mr \ln \frac{b}{a}} - \omega_o v_\phi; \quad \omega_o = \frac{eB_o}{m}$$

$$\left[ \frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r} \right] = \frac{ev_r B_o}{m} = \omega_o v_r$$

# THE MAGNETIC FIELD

$$\frac{dv_r}{dt} = \frac{dv_r}{dr} \frac{dr}{dt} = v_r \frac{dv_r}{dr} = \frac{d}{dr} \left( \frac{1}{2} v_r^2 \right)$$

$$\frac{dv_\phi}{dt} = \frac{dv_\phi}{dr} \frac{dr}{dt} = v_r \frac{dv_\phi}{dr}$$

$$v_r \left[ \frac{dv_\phi}{dr} + \frac{v_\phi}{r} \right] = \omega_o v_r \rightarrow \frac{1}{r} \frac{d}{dr} (rv_\phi) = \omega_o \rightarrow rv_\phi = \frac{\omega_o r^2}{2} + \text{constant}$$

$$v_\phi(r=a) = 0 \rightarrow v_\phi = \frac{\omega_o}{2} \left( r - \frac{a^2}{r} \right)$$

$$\frac{d}{dr} \left( \frac{1}{2} v_r^2 \right) - \frac{\omega_o^2}{4r} \left( r - \frac{a^2}{r} \right)^2 = \frac{eV_o}{m \ell n \frac{b}{a} r} - \frac{\omega_o^2}{2} \left( r - \frac{a^2}{r} \right)$$

$$\frac{d}{dr} \left( \frac{1}{2} v_r^2 \right) = \frac{eV_o}{m \ell n \frac{b}{a} r} - \frac{\omega_o^2}{4} \left( r - \frac{a^2}{r} \right) \left( 1 + \frac{a^2}{r^2} \right) = \frac{eV_o}{m \ell n \frac{b}{a} r} - \frac{\omega_o^2}{4} r \left( 1 - \frac{a^4}{r^4} \right)$$

$$\frac{1}{2} v_r^2 = \left[ \frac{eV_o}{m \ell n \frac{b}{a}} \right] \ell n \frac{r}{a} - \frac{\omega_o^2}{4} \left[ \frac{r^2}{2} + \frac{a^4}{2r^2} - a^2 \right]$$

$$= \frac{eV_o}{m \ell n \frac{b}{a}} \ell n \frac{r}{a} - \frac{\omega_o^2}{8r^2} [r^2 - a^2]^2$$

$$e) \text{ Cut-off } \rightarrow v_r(r=b) < 0 \rightarrow B_o^2 > \frac{8b^2 m V_o}{e(b^2 - a^2)^2}$$

Check: Cylindrical geometry approaches planar geometry of (a) if  $b \approx a$  so that  $b = a + s$  where  $s \ll a$ .

$$B_o^2 > \frac{8a^2 m V_o}{e 4a^2 s^2}$$

$$> \frac{2mV_o}{es^2} \quad (\text{agrees with (b)})$$

$$3. \quad m \frac{d\bar{v}}{dt} = q\bar{v} \times \bar{B} - mg\bar{i}_y \rightarrow -qv_o B_o - mg = 0 \rightarrow B_o = -\frac{mg}{qv_o}$$

$$4. \quad a) \quad \frac{1}{2} m v_x^2 - eV_1 = \frac{1}{2} m v_o^2 \rightarrow v_x^2 = v_o^2 + \frac{2e}{m} V_1$$

# THE MAGNETIC FIELD

$$b) \quad E_y - v_x B_o = 0 \rightarrow v_x = \frac{E_y}{B_o} = \sqrt{v_o^2 + \frac{2e}{m} V_1} ; \quad E_y = \frac{V_2}{s}$$

$$v_o^2 = \left(\frac{E_y}{B_o}\right)^2 - \frac{2e}{m} V_1$$

$$c) \text{ and } d) \quad R = \frac{mv_x}{eB_o} = \frac{mE_y}{eB_o^2} \rightarrow \frac{e}{m} = \frac{E_y}{RB_o^2}$$

$$5. \quad a) \quad m \frac{dv_x}{dt} = 0, \quad m \frac{dv_y}{dt} = qv_z B_x, \quad m \frac{dv_z}{dt} = -qv_y B_x$$

$$v_x = v_{xo}, \quad v_y = v_{yo} \cos \omega_o t, \quad v_z = -v_{yo} \sin \omega_o t$$

$$x = v_{xo} t, \quad y = \frac{v_{yo}}{\omega_o} \sin \omega_o t, \quad z = \frac{v_{yo}}{\omega_o} (\cos \omega_o t - 1)$$

$$y = 0 \rightarrow \omega_o t = n\pi \rightarrow \frac{\omega_o x}{v_{xo}} = n\pi \rightarrow x = \frac{n\pi v_{xo}}{\omega_o}, \quad z = \frac{-2v_{yo}}{\omega_o} \quad (n \text{ odd})$$

$$= 0 \quad (n \text{ even})$$

$$b) \quad m \frac{dv_x}{dt} = -qv_z B_o, \quad m \frac{dv_y}{dt} = 0, \quad m \frac{dv_z}{dt} = qv_x B_o$$

$$v_x = v_{xo} \cos \omega_o t, \quad v_y = v_{yo}, \quad v_z = v_{xo} \sin \omega_o t$$

$$x = \frac{v_{xo}}{\omega_o} \sin \omega_o t, \quad y = v_{yo} t, \quad z = \frac{v_{xo}}{\omega_o} (-\cos \omega_o t + 1)$$

$$y = 0 \quad (\text{never for } t > 0).$$

$$c) \quad m \frac{dv_x}{dt} = qv_y B_z, \quad m \frac{dv_y}{dt} = -qv_x B_z, \quad m \frac{dv_z}{dt} = 0$$

$$v_x = v_{yo} \sin \omega_o t + v_{xo} \cos \omega_o t, \quad v_y = v_{yo} \cos \omega_o t - v_{xo} \sin \omega_o t, \quad v_z = 0$$

$$x = \frac{1}{\omega_o} [-v_{yo} \cos \omega_o t + v_{xo} \sin \omega_o t + v_{yo}], \quad y = \frac{1}{\omega_o} [v_{yo} \sin \omega_o t + v_{xo} \cos \omega_o t - v_{xo}], \quad z = 0$$

# THE MAGNETIC FIELD

$$y = 0 \rightarrow v_{y0} \sin \omega_0 t + v_{x0} \cos \omega_0 t = v_{x0}$$

$$v_{x0} \sqrt{1 - \sin^2 \omega_0 t} = v_{x0} - v_{y0} \sin \omega_0 t$$

$$v_{x0}^2 (1 - \sin^2 \omega_0 t) = v_{x0}^2 + v_{y0}^2 \sin^2 \omega_0 t - 2v_{x0} v_{y0} \sin \omega_0 t$$

$$\sin^2 \omega_0 t (v_{x0}^2 + v_{y0}^2) - 2v_{x0} v_{y0} \sin \omega_0 t = 0$$

$$\sin \omega_0 t = 0 \text{ (one solution)} \rightarrow \omega_0 t = 2n\pi \quad n = 1, 2, \dots$$

$$\sin \omega_0 t = \frac{2v_{x0} v_{y0}}{v_{x0}^2 + v_{y0}^2}, \quad \cos \omega_0 t = \frac{(v_{x0}^2 - v_{y0}^2)}{v_{x0}^2 + v_{y0}^2}$$

$$\sin \omega_0 t = 0, \cos \omega_0 t = 1 \text{ solution} \rightarrow x = 0, y = 0, z = 0$$

Other solution for  $y = 0$

$$x = \frac{2}{\omega_0} \frac{v_{x0} v_{y0}}{v_{x0}^2 + v_{y0}^2}$$

$$6. \quad a) \quad m \frac{dv_x}{dt} = -ev_y B_0 - kx$$

$$m \frac{dv_y}{dt} = ev_x B_0 - ky$$

$$m \frac{dv_z}{dt} = -kz$$

$$b) \quad x = \hat{x}e^{st}, \quad y = \hat{y}e^{st}, \quad z = \hat{z}e^{st}; \quad \omega_0 = \frac{eB_0}{m}, \quad \omega_k = \sqrt{\frac{k}{m}}$$

$$\frac{d^2 x}{dt^2} + \omega_k^2 x = -\omega_0 \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} + \omega_k^2 y = \omega_0 \frac{dx}{dt}$$

$$\frac{d^2 z}{dt^2} + \omega_k^2 z = 0$$

$$\rightarrow \begin{bmatrix} s^2 + \omega_k^2 & \omega_0 s & 0 \\ -\omega_0 s & s^2 + \omega_k^2 & 0 \\ 0 & 0 & s^2 + \omega_k^2 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = 0$$

# THE MAGNETIC FIELD

Determinant of coefficients must be zero

$$(s^2 + \omega_k^2)[(s^2 + \omega_k^2)^2 + \omega_o^2 s^2] = 0$$

$$s^2 + \omega_k^2 = 0 \rightarrow s = \pm j\omega_k$$

$$(s^2 + \omega_k^2)^2 = -\omega_o^2 s^2 \rightarrow s^2 + \omega_k^2 = \pm j\omega_o s \rightarrow s = \pm \frac{j\omega_o}{2} \pm \sqrt{\left(\frac{\omega_o}{2}\right)^2 - \omega_k^2}$$

$$c) \quad \omega_o = \frac{eB_o}{m} \approx \frac{(1.6 \times 10^{-19})(1)}{9.1 \times 10^{-31}} \approx 1.76 \times 10^{11} \ll 10^{15} \rightarrow \omega_o \ll \omega_k$$

$$s = \pm j[\omega_k \pm \omega_o]$$

$$7. \quad \beta \bar{v} = q(\bar{E} + \bar{v} \times \bar{B})$$

$$\beta(\bar{v} \times \bar{B}) = q(\bar{E} \times \bar{B}) + q(\bar{v} \times \bar{B}) \times \bar{B}$$

$$= q(\bar{E} \times \bar{B}) + q[-\bar{v}(\bar{B} \cdot \bar{B}) + \bar{B}(\bar{v} \cdot \bar{B})]$$

$$\bar{v} \cdot \bar{B} = \frac{q}{\beta} \bar{E} \cdot \bar{B}$$

$$\beta(\bar{v} \times \bar{B}) = q(\bar{E} \times \bar{B}) - q\bar{v}(\bar{B} \cdot \bar{B}) + \frac{q^2}{\beta} \bar{B}(\bar{E} \cdot \bar{B}) = \beta \left[ \frac{\beta}{q} \bar{v} - \bar{E} \right]$$

$$\bar{v} \left[ \frac{\beta^2}{q} + q(\bar{B} \cdot \bar{B}) \right] = q \left[ \bar{E} \times \bar{B} + \frac{q\bar{B}}{\beta} (\bar{E} \cdot \bar{B}) + \frac{\beta\bar{E}}{q} \right]$$

$$\bar{v} = \frac{\bar{E} \times \bar{B} + \frac{q\bar{B}}{\beta} (\bar{E} \cdot \bar{B}) + \frac{\beta\bar{E}}{q}}{\frac{\beta^2}{q} + \bar{B} \cdot \bar{B}}$$

$$8. \quad a) \quad \bar{m} \frac{d\bar{v}}{dt} = -m\bar{v} + q(\bar{E} + \bar{v} \times \bar{B}) - \frac{1}{n} \nabla(nkT)$$

$$b) \quad \bar{v} \approx \frac{q}{m\bar{v}} (\bar{E} + \bar{v} \times \bar{B})$$

$$c) \quad \bar{J} = qn\bar{v} = \frac{q^2 n}{m\bar{v}} (\bar{E} + \bar{v} \times \bar{B}) = \sigma(\bar{E} + \bar{v} \times \bar{B})$$

# THE MAGNETIC FIELD

$$d) \quad J_x = \sigma \left( -\frac{v_h}{d} + v_y B_o \right) = \frac{i_h}{A} = \frac{v_h}{R_L A}$$

$$v_h \left[ \frac{\sigma}{d} + \frac{1}{R_L A} \right] = \sigma v_y B_o$$

$$v_h = \frac{\sigma v_y B_o A}{\frac{1}{R_L} + \frac{1}{R_{int}}} ; \quad R_{int} = \frac{d}{\sigma A}$$

$$e) \quad P = \frac{v_h^2}{R_L} = \frac{(\sigma v_y B_o A) R_{int}^2 R_L}{(R_L + R_{int})^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{(\sigma v_y B_o A) R_{int}^2}{(R_L + R_{int})^2} \left[ 1 - \frac{2R_L}{R_L + R_{int}} \right] = 0 \rightarrow R_L = R_{int}$$

## Section 5.2

$$9. \quad a) \quad \vec{B} = \frac{\mu_o I}{2\pi r} \vec{i}_\phi$$

$$\frac{m d\vec{v}}{dt} = q\vec{v} \times \vec{B}$$

$$\frac{dv_r}{dt} - \frac{v_\phi^2}{r} = \frac{d}{dr} \left( \frac{1}{2} v_r^2 \right) - \frac{v_\phi^2}{r} = -\frac{q}{m} v_z B_\phi$$

$$\frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r} = v_r \left[ \frac{dv_\phi}{dr} + \frac{v_\phi}{r} \right] = 0 \rightarrow r v_\phi = r_o v_{\phi o}$$

$$\frac{dv_z}{dt} = v_r \frac{dv_z}{dr} = \frac{q}{m} v_r B_\phi \rightarrow \frac{dv_z}{dr} = \frac{q \mu_o I}{m 2\pi r} \rightarrow v_z = \frac{q \mu_o I}{2\pi m} \ln \frac{r}{r_o} + v_{zo}$$

$$\frac{d}{dr} \left( \frac{1}{2} v_r^2 \right) = \frac{v_\phi^2}{r} - \frac{q}{m} v_z B_\phi = \frac{(r_o v_{\phi o})^2}{r^3} - \frac{q \mu_o I}{m 2\pi r} \left[ \frac{q \mu_o I}{2\pi m} \ln \frac{r}{r_o} + v_{zo} \right]$$

$$\frac{1}{2} v_r^2 = -\frac{1}{2} (r_o v_{\phi o})^2 \left( \frac{1}{r^2} - \frac{1}{r_o^2} \right) - \frac{q \mu_o I v_{zo}}{2\pi m} \ln \frac{r}{r_o} - \frac{1}{2} \left( \frac{q \mu_o I}{2\pi m} \right)^2 \left( \ln \frac{r}{r_o} \right)^2 + \frac{1}{2} v_{ro}^2$$

# THE MAGNETIC FIELD

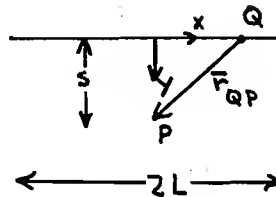
$$\begin{aligned} \text{b) } K.E. &= \frac{1}{2} m [v_r^2 + v_\phi^2 + v_z^2] \\ &= \frac{1}{2} m [v_{ro}^2 + v_{\phi o}^2 + v_{zo}^2] \end{aligned}$$

$$\text{c) With } v_{\phi o} = 0, \text{ for } v_r = 0$$

$$\left[ \frac{q\mu_o I}{2\pi m} \ln\left(\frac{r}{r_o}\right) \right]^2 + 2v_{zo} \left[ \frac{q\mu_o I}{2\pi m} \ln\left(\frac{r}{r_o}\right) \right] - v_{ro}^2 = 0$$

$$\frac{q\mu_o I}{2\pi m} \ln\left(\frac{r}{r_o}\right) = -v_{zo} \pm \sqrt{v_{zo}^2 + v_{ro}^2}$$

$$10. \quad \vec{B} = \frac{\mu_o}{4\pi} \int_L \frac{\vec{Idl} \times \vec{i}_{QP}}{r_{QP}^2}$$



$$\vec{r}_{QP} = -(x\vec{i}_x - s\vec{i}_y)$$

$$\vec{i}_{QP} = -\frac{(x\vec{i}_x - s\vec{i}_y)}{\sqrt{x^2 + s^2}}$$

$$\vec{B} = \frac{\mu_o I s}{4\pi} \vec{i}_z \int_{-L}^{+L} \frac{dx}{(x^2 + s^2)^{3/2}}$$

$$= \frac{\mu_o I s}{4\pi} \vec{i}_z \left. \frac{x}{s^2 [x^2 + s^2]^{1/2}} \right|_{-L}^L$$

$$= \frac{\mu_o I L}{2\pi s [L^2 + s^2]^{1/2}} \vec{i}_z$$

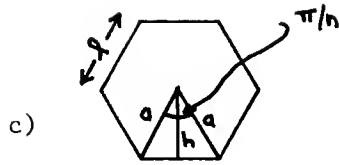
a) Let  $L \rightarrow a$ ,  $L \rightarrow b$  and use superposition

$$\vec{B} = \frac{2\mu_o I}{2\pi} \left\{ \frac{a/2}{[(\frac{a}{2})^2 + (\frac{b}{2})^2]^{1/2} \frac{b}{2}} + \frac{b/2}{[(\frac{a}{2})^2 + (\frac{b}{2})^2]^{1/2} \frac{a}{2}} \right\} \vec{i}_z$$

$$= \frac{2\mu_o I}{\pi [a^2 + b^2]^{1/2}} \left[ \frac{a}{b} + \frac{b}{a} \right] \vec{i}_z = \frac{2\mu_o I [a^2 + b^2]^{1/2}}{\pi ab} \vec{i}_z$$

$$\begin{aligned} \text{b) } \vec{B} &= -\frac{\mu_o I a}{4\pi} \int_0^{2\pi} \frac{\vec{i}_\phi \times \vec{i}_r}{a^2} d\phi \\ &= \frac{\mu_o I}{2a} \vec{i}_z \end{aligned}$$

# THE MAGNETIC FIELD



$$h^2 = a^2 - \frac{l^2}{4} = a^2 \cos^2 \frac{\pi}{n} \rightarrow \left(\frac{l}{2}\right)^2 = a^2 (1 - \cos^2 \frac{\pi}{n}) = a^2 \sin^2 \frac{\pi}{n}$$

$$B_z = \frac{n \mu_o I \frac{l}{2}}{2\pi h \left[ \left(\frac{l}{2}\right)^2 + h^2 \right]^{1/2}}$$

$$= \frac{n \mu_o I \sin \frac{\pi}{n}}{2\pi a \cos \frac{\pi}{n}} = \frac{n \mu_o I}{2\pi a} \tan \frac{\pi}{n}$$

Check:  $\lim_{n \rightarrow \infty} B_z = \frac{\mu_o I}{2a}$  (agrees with (b))

$\lim_{n \rightarrow 4} B_z = \frac{2\mu_o I}{\pi a}$  (agrees with (a) when  $a = b$  and  $a$  (of (c))  $\rightarrow \sqrt{2}a$  (of (a)))

d) No contribution from straight line segments

$$\vec{B} = \frac{\mu_o I}{4\pi} \int_{\phi=0}^{\pi} \frac{a \vec{i}_{\phi} \times (-\vec{i}_r) d\phi}{a^2}$$

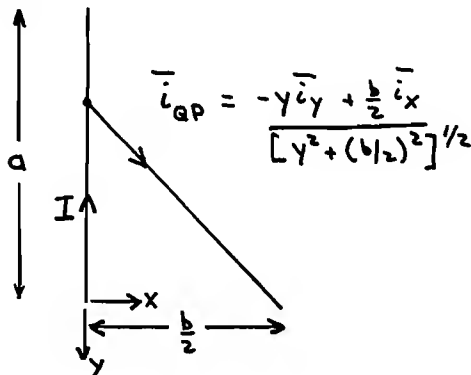
$$= \frac{\mu_o I}{4a} \vec{i}_z$$

e) No contribution from semi-infinite line currents.

From horizontal length

$$B_z = \frac{\mu_o I \frac{b}{2}}{2\pi a \left[ \left(\frac{b}{2}\right)^2 + a^2 \right]^{1/2}}$$

From vertical length



$$\vec{B} = \frac{\mu_o I}{4\pi} \int_{-a}^0 \frac{-\vec{i}_y \times \vec{i}_{QP} dy}{[y^2 + (\frac{b}{2})^2]}$$

$$= \frac{\mu_o I \vec{i}_z}{4\pi} \int_{-a}^0 \frac{\frac{b}{2} dy}{[y^2 + (\frac{b}{2})^2]^{3/2}}$$

$$= \frac{\mu_o I}{4\pi} \vec{i}_z \frac{b}{2} \frac{y}{(\frac{b}{2})^2 [y^2 + (\frac{b}{2})^2]^{1/2}} \bigg|_{y=-a}^0$$



# THE MAGNETIC FIELD

$$= \frac{\mu_o I a \bar{i}_z}{2\pi b [a^2 + (\frac{b}{2})^2]^{1/2}}$$

$$B_{ztotal} = \frac{\mu_o I}{2\pi [a^2 + (\frac{b}{2})^2]^{1/2}} [\frac{2a}{b} + \frac{b}{2a}] = \frac{\mu_o I}{\pi ab} [a^2 + (\frac{b}{2})^2]^{1/2}$$

f) From straight line segments,  $B_z = \frac{\mu_o I}{2\pi a}$  (see (e))

From semi-circle,  $B_z = \frac{\mu_o I}{4a}$  (see (d))

$$B_{ztotal} = \frac{\mu_o I}{2a} (\frac{1}{2} + \frac{1}{\pi})$$

11.  $F_r = \frac{\mu_o I^2}{2\pi(2\ell \sin \frac{\theta}{2})} = T \sin \frac{\theta}{2}, \quad T \cos \frac{\theta}{2} = mg$

$$T \sin^2 \frac{\theta}{2} = \frac{\mu_o I^2}{4\pi \ell} = \frac{mg \sin^2 \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{mg \sin^2 \frac{\theta}{2}}{\sqrt{1 - \sin^2 \frac{\theta}{2}}}$$

$$\sin^4 \frac{\theta}{2} + (\frac{\mu_o I^2}{4\pi \ell mg})^2 \sin^2 \frac{\theta}{2} - (\frac{\mu_o I^2}{4\pi \ell mg})^2 = 0$$

$$\sin^2 \frac{\theta}{2} = -\frac{1}{2} (\frac{\mu_o I^2}{4\pi \ell mg})^2 \pm \sqrt{[\frac{1}{2} (\frac{\mu_o I^2}{4\pi \ell mg})^2]^2 + (\frac{\mu_o I^2}{4\pi \ell mg})^2}$$

$$= \frac{1}{2} (\frac{\mu_o I^2}{4\pi \ell mg})^2 \left[ -1 + \sqrt{1 + 4 \left[ \frac{4\pi \ell mg}{\mu_o I^2} \right]^2} \right]$$

12. a)  $\overline{dB} = -\frac{\mu_o}{4\pi} K_o \bar{i}_\phi \times \bar{i}_r \sin \theta d\theta d\phi$

$$= -\frac{\mu_o K_o}{4\pi} \sin \theta d\theta d\phi [\bar{i}_x \cos \theta \cos \phi + \cos \theta \sin \phi \bar{i}_y - \sin \theta \bar{i}_z]$$

$$\bar{B} = -\frac{\mu_o K_o}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin \theta d\theta d\phi [\bar{i}_x \cos \theta \cos \phi + \cos \theta \sin \phi \bar{i}_y - \sin \theta \bar{i}_z]$$

$$= \frac{\mu_o K_o}{2} \int_{\theta=0}^{\pi} \sin^2 \theta d\theta \bar{i}_z$$

# THE MAGNETIC FIELD

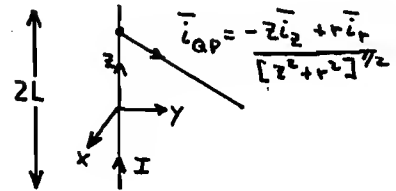
$$= \frac{\mu_0 K_0 \pi}{4} \bar{i}_z$$

b)  $K_0 \rightarrow J_0 dr$

$$B_z = \frac{\mu_0 J_0 \pi}{4} \int_a^b dr = \frac{\mu_0 J_0 \pi (b - a)}{4}$$

13. a) 
$$\vec{dB} = \frac{\mu_0 I}{4\pi} dz \frac{(\bar{i}_z \times \bar{i}_{QP})}{(z^2 + r^2)}$$

$$= \frac{\mu_0 I}{4\pi (z^2 + r^2)^{3/2}} r dz \bar{i}_\phi$$



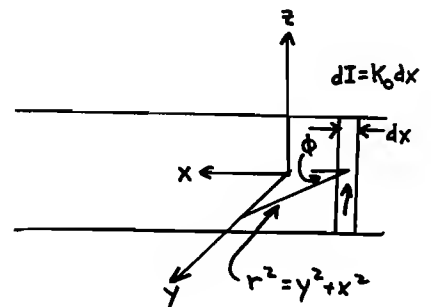
$$B_\phi = \frac{\mu_0 I}{4\pi} \int_{z=-L}^L \frac{dz}{(z^2 + r^2)^{3/2}} = \frac{\mu_0 I r}{4\pi} \left. \frac{z}{r^2 \sqrt{z^2 + r^2}} \right|_{z=-L}^L = \frac{\mu_0 I L}{2\pi r \sqrt{L^2 + r^2}}$$

b)  $B_y = 0$

$$dB_x = \frac{\mu_0 L K_0 dx}{2\pi \sqrt{y^2 + x^2} \sqrt{L^2 + y^2 + x^2}} \sin\phi ; \sin\phi = \frac{y}{\sqrt{y^2 + x^2}}$$

$$= \frac{\mu_0 L K_0 y dx}{2\pi (y^2 + x^2) [L^2 + y^2 + x^2]^{1/2}}$$

$$B_x = \frac{\mu_0 L K_0 y}{2\pi} \int_{x=-\infty}^{+\infty} \frac{dx}{(y^2 + x^2) (L^2 + y^2 + x^2)^{1/2}}$$



Let  $u = x^2 + y^2$ ,  $du = 2x dx$ ,  $x = \sqrt{u - y^2} \rightarrow du = 2\sqrt{u - y^2} dx$   $y > 0$

$$B_x = \frac{\mu_0 L K_0 y}{2\pi} \int_{y^2}^{\infty} \frac{du}{\sqrt{u - y^2} u \sqrt{L^2 + u}}$$

$$= \frac{\mu_0 K_0 L y}{2\pi} \int_{y^2}^{\infty} \frac{du}{u [u^2 + u(L^2 - y^2) - y^2 L^2]^{1/2}}$$

$$= \frac{\mu_0 K_0 L y}{2\pi} \frac{1}{yL} \sin^{-1} \left[ \frac{(L^2 - y^2)u - 2y^2 L^2}{u \sqrt{(L^2 - y^2)^2 + 4y^2 L^2}} \right] \Big|_{u=y^2}^{\infty}$$

# THE MAGNETIC FIELD

$$= \frac{\mu_o K_o}{2\pi} \left\{ \sin^{-1} \frac{(L^2 - y^2)}{(L^2 + y^2)} - \sin^{-1}(-1) \right\}$$

$$= \frac{\mu_o K_o}{2\pi} \left\{ \frac{\pi}{2} + \sin^{-1} \frac{(L^2 - y^2)}{(L^2 + y^2)} \right\} \quad ; y > 0$$

Check:

$$\lim_{L \rightarrow \infty} B_x = \frac{\mu_o K_o}{2}$$

14. Use superposition

$$\vec{B}_1 = \begin{cases} \mu_o K_o \vec{i}_z & r < a \\ 0 & r > a \end{cases} ; \quad \vec{B}_2 = \begin{cases} 0 & r < a \\ \frac{\mu_o K_o a}{r} \vec{i}_\phi & r > a \end{cases}$$

$$\vec{B}_T = \vec{B}_1 + \vec{B}_2 = \begin{cases} \mu_o K_o \cos\theta_o \vec{i}_z & r < a \\ \frac{\mu_o K_o a}{r} \sin\theta_o \vec{i}_\phi & r > a \end{cases}$$

15. a) Replace hole by currents  $J_o \vec{i}_z$  and  $-J_o \vec{i}_z$  and use superposition.

Field due to slab with uniform current is given by (12) in (5.2.3b).

$$\vec{B}_1 = \begin{cases} -\frac{\mu_o J_o d}{2} \vec{i}_x & y > \frac{d}{2} \\ -\mu_o J_o y \vec{i}_x & |y| < \frac{d}{2} \\ \frac{\mu_o J_o d}{2} \vec{i}_x & y < -\frac{d}{2} \end{cases}$$

Field due to hole with current  $-J_o \vec{i}_z$  [from (23) in (5.3.3b)]

$$\vec{B}_2 = \begin{cases} -\frac{\mu_o J_o a^2}{2r} \vec{i}_\phi = -\frac{\mu_o J_o a^2}{2[x^2 + y^2]^{1/2}} (-\sin\phi \vec{i}_x + \cos\phi \vec{i}_y) \\ \quad = -\frac{\mu_o J_o a^2}{2(x^2 + y^2)} [-y \vec{i}_x + x \vec{i}_y] & r > a \\ -\frac{\mu_o J_o r}{2} \vec{i}_\phi = -\frac{\mu_o J_o}{2} [x^2 + y^2]^{1/2} [-\sin\phi \vec{i}_x + \cos\phi \vec{i}_y] \end{cases}$$

# THE MAGNETIC FIELD

$$= -\frac{\mu_o J_o}{2} [-y\bar{i}_x + x\bar{i}_y] \quad r < a$$

$$\bar{B}_T = \bar{B}_1 + \bar{B}_2$$

b) Fill hole with current  $J_o \bar{i}_z$

$$\bar{B}_1 = \frac{\mu_o J_o r}{2} \bar{i}_\phi = -\frac{\mu_o J_o}{2} [y\bar{i}_x - x\bar{i}_y] \quad (\text{in hole})$$

Fill hole with current  $-J_o \bar{i}_z$

$$\bar{B}_2 = \frac{\mu_o J_o}{2} [y\bar{i}_x - (x-d)\bar{i}_y] \quad (\text{in hole})$$

$$\bar{B}_T = \bar{B}_1 + \bar{B}_2 = \frac{\mu_o J_o d}{2} \bar{i}_y$$

## Section 5.3

16. a)  $\bar{B} = ar\bar{i}_r$

$$\nabla \cdot \bar{B} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 B_r) = 3a \neq 0 \quad \text{Not a magnetic field}$$

b)  $\bar{B} = a(x\bar{i}_y - y\bar{i}_x)$

$$\nabla \cdot \bar{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \quad \text{Can be a magnetic field}$$

$$\nabla \times \bar{B} = \bar{i}_z \left[ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] = 2a\bar{i}_z = \mu_o \bar{J}$$

c)  $\bar{B} = a(x\bar{i}_x - y\bar{i}_y)$

$$\nabla \cdot \bar{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \quad \text{Can be a magnetic field}$$

$$\nabla \times \bar{B} = \bar{i}_z \left[ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] = \mu_o \bar{J} = 0$$

# THE MAGNETIC FIELD

$$d) \quad \vec{B} = a r \vec{i}_\phi$$

$$\nabla \cdot \vec{B} = \frac{1}{r} \frac{\partial B_\phi}{\partial \phi} = 0 \quad \text{Can be a magnetic field}$$

$$\nabla \times \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi) \vec{i}_z = 2a \vec{i}_z = \mu_0 \vec{J}$$

$$17. \quad a) \quad \frac{\partial B_x}{\partial y} = \begin{cases} -\mu_0 J_0 & -a < y < 0 \\ \mu_0 J_0 & 0 < y < a \end{cases}$$

$$B_x = \begin{cases} -\mu_0 J_0 (y + a) & -a < y < 0 \\ \mu_0 J_0 (y - a) & 0 < y < a \\ 0 & |y| > a \end{cases}$$

$$b) \quad \frac{\partial B_x}{\partial y} = \frac{-\mu_0 J_0 y}{a} \rightarrow B_x = \begin{cases} \frac{-\mu_0 J_0}{2a} (y^2 - a^2) & |y| < a \\ 0 & |y| > a \end{cases}$$

$$c) \quad \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi) = \pm \mu_0 J_0 \rightarrow r B_\phi = \pm \frac{\mu_0 J_0 r^2}{2} + \text{constant}$$

$$B_\phi = \begin{cases} \frac{\mu_0 J_0 r}{2} & 0 < r < a \\ -\frac{\mu_0 J_0 r}{2} \left(1 - \frac{2a^2}{r^2}\right) & a < r < b \\ \frac{\mu_0 J_0}{r} \left(a^2 - \frac{b^2}{2}\right) & r > b \end{cases}$$

$$d) \quad \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi) = \frac{\mu_0 J_0 r}{a} \rightarrow r B_\phi = \frac{\mu_0 J_0 r^3}{3a} + \text{constant}$$

$$B_\phi = \begin{cases} \frac{\mu_0 J_0 r^2}{3a} & r < a \\ \frac{\mu_0 J_0 a^2}{3r} & r > a \end{cases}$$

# THE MAGNETIC FIELD

## Section 5.4

$$\begin{aligned}
 18. \quad a) \quad A_z &= \frac{-\mu_0 K_0}{4\pi} \left\{ \int_{-\infty}^0 \{ \ln[(x'-x)^2 + (y-d)^2] - \ln[(x-x')^2 + (y+d)^2] \} dx' \right\} \\
 &= \frac{-\mu_0 K_0}{4\pi} \left\{ (x'-x) \ln[(x'-x)^2 + (y-d)^2] - 2(x'-x) + 2(y-d) \tan^{-1} \left( \frac{x'-x}{y-d} \right) \right. \\
 &\quad \left. - (x'-x) \ln[(x'-x)^2 + (y+d)^2] + 2(x'-x) - 2(y+d) \tan^{-1} \left( \frac{x'-x}{y+d} \right) \right\} \Bigg|_{x'=-\infty}^0 \\
 &= \frac{-\mu_0 K_0}{4\pi} \left\{ -x \{ \ln[x^2 + (y-d)^2] - \ln[x^2 + (y+d)^2] \} \right. \\
 &\quad \left. - 2(y-d) \left[ \tan^{-1} \frac{x}{y-d} + \frac{\pi}{2} \right] \right. \\
 &\quad \left. + 2(y+d) \left[ \tan^{-1} \frac{x}{y+d} - \frac{\pi}{2} \right] \right\} \quad |y| < d
 \end{aligned}$$

$$b) \quad \vec{B} = \nabla \times \vec{A} = \vec{i}_x \frac{\partial A_z}{\partial y} - \vec{i}_y \frac{\partial A_z}{\partial x}$$

$$\begin{aligned}
 B_x &= \frac{\partial A_z}{\partial y} = \frac{-\mu_0 K_0}{4\pi} \left\{ -x \left[ \frac{2(y-d)}{x^2 + (y-d)^2} - \frac{2(y+d)}{x^2 + (y+d)^2} \right] + 2 \left[ -\tan^{-1} \frac{x}{y-d} - \pi + \tan^{-1} \frac{x}{y+d} \right] \right. \\
 &\quad \left. + 2(y-d) \frac{x}{x^2 + (y-d)^2} - \frac{2(y+d)x}{(y+d)^2 + x^2} \right\} \\
 &= \frac{\mu_0 K_0}{2\pi} \left[ \tan^{-1} \frac{x}{y-d} - \tan^{-1} \frac{x}{y+d} + \pi \right]
 \end{aligned}$$

$$\begin{aligned}
 B_y &= -\frac{\partial A_z}{\partial x} = \frac{\mu_0 K_0}{4\pi} \left\{ -\ln[x^2 + (y-d)^2] + \ln[x^2 + (y+d)^2] - 2x^2 \left[ \frac{1}{x^2 + (y-d)^2} - \frac{1}{x^2 + (y+d)^2} \right] \right. \\
 &\quad \left. - \frac{2(y-d)^2}{x^2 + (y-d)^2} + \frac{(y+d)^2}{x^2 + (y+d)^2} \right\} \\
 &= \frac{-\mu_0 K_0}{4\pi} \ln \frac{x^2 + (y-d)^2}{x^2 + (y+d)^2}
 \end{aligned}$$

$$c) \quad \Phi = \int_{y=-d}^{+d} B_x dy = \int_{y=-d}^{+d} \frac{\partial A_z}{\partial y} dy = [A_z(y=d) - A_z(y=-d)] \Bigg|_{x=0}$$

# THE MAGNETIC FIELD

$$A_z(x=0, y) = \frac{\mu_o K_o y}{2} \rightarrow \Phi = \mu_o K_o d$$

$$\Phi(x=-\infty) = \mu_o K_o (2d) \rightarrow \Phi \text{ through each current sheet} = \frac{\mu_o K_o d}{2}$$

$$d) A_z(x=0, y=y_o) = \frac{\mu_o K_o y_o}{2}$$

$$A_z(x=-\infty, y) = \mu_o K_o y \rightarrow A_z(x=-\infty, y) = A_z(x=0, y=y_o) \rightarrow y = \frac{y_o}{2} \text{ at } x = -\infty$$

$$19. a) A_z = \frac{\mu_o I}{4\pi} \left\{ \sinh^{-1} \frac{\frac{L}{2} - z}{r} + \sinh^{-1} \frac{\frac{L}{2} + z}{r} \right\}$$

$$\nabla \cdot \bar{A} = \frac{\partial A_z}{\partial z} = \frac{\mu_o I}{4\pi} \left\{ \frac{-1}{\sqrt{r^2 + (\frac{L}{2} - z)^2}} + \frac{1}{\sqrt{r^2 + (\frac{L}{2} + z)^2}} \right\} \neq 0$$

b) For a square loop

$$\begin{aligned} \bar{A} = \frac{\mu_o I}{4\pi} \left\{ \bar{i}_z \left[ \sinh^{-1} \frac{\frac{L}{2} - z}{[x^2 + (y + \frac{L}{2})^2]^{1/2}} + \sinh^{-1} \frac{\frac{L}{2} + z}{[x^2 + (y + \frac{L}{2})^2]^{1/2}} - \sinh^{-1} \frac{\frac{L}{2} - z}{[x^2 + (y - \frac{L}{2})^2]^{1/2}} \right. \right. \\ \left. \left. - \sinh^{-1} \frac{\frac{L}{2} + z}{[x^2 + (y - \frac{L}{2})^2]^{1/2}} \right] + \bar{i}_y \left[ \sinh^{-1} \frac{\frac{L}{2} - y}{[x^2 + (z - \frac{L}{2})^2]^{1/2}} \right. \right. \\ \left. \left. + \sinh^{-1} \frac{\frac{L}{2} + y}{[x^2 + (z - \frac{L}{2})^2]^{1/2}} - \sinh^{-1} \frac{\frac{L}{2} - y}{[x^2 + (z + \frac{L}{2})^2]^{1/2}} - \sinh^{-1} \frac{\frac{L}{2} + y}{[x^2 + (z + \frac{L}{2})^2]^{1/2}} \right] \right\} \end{aligned}$$

$$c) \nabla \cdot \bar{A} = \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned} = \frac{\mu_o I}{4\pi} \left\{ \frac{-1}{[x^2 + (y + \frac{L}{2})^2 + (\frac{L}{2} - z)^2]^{1/2}} + \frac{1}{[x^2 + (y + \frac{L}{2})^2 + (\frac{L}{2} + z)^2]^{1/2}} + \frac{1}{[x^2 + (y - \frac{L}{2})^2 + (\frac{L}{2} - z)^2]^{1/2}} \right. \\ \left. - \frac{1}{[x^2 + (y - \frac{L}{2})^2 + (\frac{L}{2} + z)^2]^{1/2}} - \frac{1}{[x^2 + (z - \frac{L}{2})^2 + (\frac{L}{2} - y)^2]^{1/2}} + \frac{1}{[x^2 + (z - \frac{L}{2})^2 + (\frac{L}{2} + y)^2]^{1/2}} \right\} \end{aligned}$$

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$$+ \frac{1}{[x^2 + (z + \frac{L}{2})^2 + (\frac{L}{2} - y)^2]^{1/2}} - \frac{1}{[x^2 + (z + \frac{L}{2})^2 + (\frac{L}{2} + y)^2]^{1/2}} \}$$

$$= 0$$

$$20. \quad \bar{A} = \frac{\mu_0}{4\pi} \int_S \frac{\bar{K} dS}{r_{QP}}, \quad \nabla^2 \bar{A} = -\mu_0 \bar{J} \rightarrow \nabla \times (\nabla \times \bar{A}) = \mu_0 \bar{J}$$

$$a) \quad \bar{A} = A_\phi(r) \bar{i}_\phi, \quad \nabla \times \bar{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \bar{i}_z$$

$$\nabla \times (\nabla \times \bar{A}) = -\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \right] \bar{i}_\phi = 0 \rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) = C$$

$$r A_\phi = \frac{C r^2}{2} + D$$

$$A_\phi = \begin{cases} \frac{C_1 r}{2} & 0 < r < a \\ \frac{D}{r} & r > a \end{cases}$$

$$\bar{B} = \nabla \times \bar{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \bar{i}_z = \begin{cases} C_1 \bar{i}_z = \mu_0 K_0 \bar{i}_z & 0 < r < a \\ 0 & r > a \end{cases}$$

$$b) \text{ and } c) \quad \bar{A} = A_z(r) \bar{i}_z$$

$$\nabla \times \bar{A} = -\frac{\partial A_z}{\partial r} \bar{i}_\phi, \quad \nabla \times (\nabla \times \bar{A}) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_z}{\partial r} \right) = -\mu_0 J_0$$

$$r \frac{\partial A_z}{\partial r} = \frac{-\mu_0 J_0 r^2}{2} + C \rightarrow A_z = \frac{-\mu_0 J_0 r^2}{4} + C \ln r + D$$

$$b) \quad \text{Surface current } K_0 \bar{i}_z \rightarrow J_0 = 0$$

$$A_z = \begin{cases} D_1 & 0 < r < a \\ C \ln r + D_2 & r > a \end{cases}$$

$$\bar{B} = \nabla \times \bar{A} = -\frac{\partial A_z}{\partial r} \bar{i}_\phi = \begin{cases} 0 & 0 < r < a \\ -\frac{C}{r} \bar{i}_\phi = \frac{\mu_0 K_0 a}{r} \bar{i}_\phi & r > a \end{cases}$$



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c) Volume current  $J_o \bar{i}_z$

$$A_z = \begin{cases} \frac{-\mu_o J_o r^2}{4} + D_1 \\ C \ln r + D_2 \end{cases}; \quad B_\phi = -\frac{\partial A_z}{\partial r} = \begin{cases} \frac{\mu_o J_o r}{2} \\ \frac{\mu_o J_o a^2}{2r} \end{cases} \quad \begin{matrix} 0 < r < a \\ r > a \end{matrix}$$

d)  $\bar{A} = A_z(x) \bar{i}_z$

$$\frac{\partial^2 A_z}{\partial x^2} = -\mu_o J_o \rightarrow A_z = \frac{-\mu_o J_o x^2}{2} + A_1 x + A_2$$

$$\bar{B} = \nabla \times \bar{A} = -\bar{i}_y \frac{\partial A_z}{\partial x} = (\mu_o J_o x - A_1) \bar{i}_y$$

$$\bar{B} = \begin{cases} \mu_o J_o x & |y| < \frac{d}{2} \\ \mu_o J_o \frac{d}{2} & y > \frac{d}{2} \\ -\mu_o J_o \frac{d}{2} & y < -\frac{d}{2} \end{cases}$$

e)  $\frac{\partial^2 A_z}{\partial x^2} = \frac{-\mu_o J_o x}{d} \rightarrow A_z = \frac{-\mu_o J_o x^3}{6d} + A_1 x + A_2$

$$\bar{B} = \nabla \times \bar{A} = -\bar{i}_y \frac{\partial A_z}{\partial x} = \left( \frac{\mu_o J_o x^2}{2d} - A_1 \right) \bar{i}_y \rightarrow B_y = \begin{cases} \frac{\mu_o J_o}{2d} \left( x^2 - \frac{d^2}{2} \right) & |x| < \frac{d}{2} \\ 0 & |x| > \frac{d}{2} \end{cases}$$

## Section 5.5

21. a)  $\bar{m} = \frac{1}{2} \bar{r} \times q \bar{v} = \frac{1}{2} q \omega a^2 (\bar{i}_r \times \bar{i}_\phi) = \frac{1}{2} q \omega a^2 \bar{i}_z$

b)  $\bar{m} = \frac{1}{2} \oint a \bar{i}_r \times I a d\phi \bar{i}_\phi$   
 $= I \pi a^2 \bar{i}_z$

c)  $\bar{m} = \frac{1}{2} \int_{\phi=0}^{2\pi} \int_{r=0}^a r K_o (\bar{i}_r \times \bar{i}_\phi) r dr d\phi$

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$$= K_o \pi \bar{i}_z \int_{r=0}^a r^2 dr$$

$$= \frac{K_o \pi a^3}{3} \bar{i}_z$$

d) Surface charge  $\sigma_f = \frac{Q}{4\pi R^2}$ ;  $\bar{K} = \sigma_f \omega R \bar{i}_\phi = \frac{Q\omega}{4\pi R} \bar{i}_\phi$

$$\bar{m} = \frac{1}{2} \frac{Q\omega}{4\pi R} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} R^3 (\underbrace{\bar{i}_r \times \bar{i}_\phi}_{-\bar{i}_\theta}) \sin\theta d\theta d\phi$$

$$= -\frac{1}{2} \frac{Q\omega R^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} [\cos\theta \cos\phi \bar{i}_x + \cos\theta \sin\phi \bar{i}_y - \sin\theta \bar{i}_z] \sin\theta d\theta d\phi$$

$m_x = 0, m_y = 0$  when integrated over  $\phi$

$$m_z = \frac{1}{4} Q\omega R^2 \int_{\theta=0}^{\pi} \sin^2\theta d\theta$$

$$= Q\omega R^2 \frac{\pi}{8}$$

Volume charge  $\rho_f = \frac{Q}{\frac{4}{3}\pi R^3}$ ;  $\bar{J} = \rho_f \omega r \bar{i}_\phi = \frac{Q\omega}{\frac{4}{3}\pi R^3} r \bar{i}_\phi$

$$\bar{m} = \frac{1}{2} \frac{Q\omega}{\frac{4}{3}\pi R^3} \int_{r=0}^R \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 (\bar{i}_r \times \bar{i}_\phi) \sin\theta dr d\theta d\phi$$

$$m_z = \frac{3Q\omega}{4R^3} \int_{r=0}^R \int_{\theta=0}^{\pi} r^4 \sin^2\theta dr d\theta$$

$$= \frac{3Q\omega\pi R^2}{40}$$

22. a)  $\bar{H} = [H_o + \frac{m}{2\pi|z|^3}] \bar{i}_z$  along z axis

b)  $f_z = \mu_o m_z \frac{\partial H_z}{\partial z} = \mu_o m [\frac{-3m}{2\pi z^4}] = -\frac{3\mu_o m^2}{2\pi z^4}$

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$$c) \quad H_z = H_o + \frac{m}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(na)^3} \approx H_o + \frac{1.2m}{2\pi a^3} \approx H_o + \frac{1.2}{2\pi a^3} \alpha H_z$$

$$H_z \approx \frac{H_o}{1 - \frac{1.2\alpha}{2\pi a^3}}, \quad m_z = \alpha H_z = \frac{\alpha H_o}{1 - \frac{1.2\alpha}{2\pi a^3}}$$

$$d) \quad B_z = \mu_o (H_o + \frac{m_z}{a^3})$$

$$= \mu_o H_o \left[ 1 + \frac{\alpha}{a^3 \left[ 1 - \frac{1.2\alpha}{2\pi a^3} \right]} \right] \rightarrow \mu = \mu_o \left[ 1 + \frac{\alpha}{a^3 \left[ 1 - \frac{1.2\alpha}{2\pi a^3} \right]} \right]$$

$$23. \quad a) \quad \frac{d\vec{L}}{dt} = \vec{m} \times \vec{B}, \quad \vec{L} = -\frac{2m_e}{e} \vec{m} \rightarrow \frac{d\vec{m}}{dt} = -\gamma \vec{m} \times \vec{B}; \quad \gamma = \frac{e}{2m_e}$$

$$b) \quad \frac{dm_x}{dt} = -\gamma m_y B_o, \quad \frac{dm_y}{dt} = \gamma m_x B_o, \quad \frac{dm_z}{dt} = 0$$

$$\frac{d^2 m_x}{dt^2} + \omega_o^2 m_x = 0; \quad \omega_o^2 = \gamma B_o$$

$$m_x = m_{yo} \sin \omega_o t + m_{xo} \cos \omega_o t$$

$$m_y = m_{yo} \cos \omega_o t - m_{xo} \sin \omega_o t$$

$$m_z = m_{zo}$$

$$c) \quad \text{Precessional frequency} = \omega_o.$$

$$24. \quad a) \quad \oint_L \vec{H} \cdot d\vec{\ell} = \int_S \vec{J}_f \cdot d\vec{S} \rightarrow H_\phi (2\pi r) = \begin{cases} J_o \pi r^2 & r < a \\ J_o \pi a^2 & r > a \end{cases}$$

$$H_\phi = \begin{cases} \frac{J_o r}{2} \\ \frac{J_o a^2}{2r} \end{cases}; \quad B_\phi = \begin{cases} \frac{\mu J_o r}{2} \\ \frac{\mu_o J_o a^2}{2r} \end{cases}; \quad M_\phi = \frac{B_\phi}{\mu_o} - H_\phi = \begin{cases} \frac{(\mu - \mu_o) J_o r}{2\mu_o} & r < a \\ 0 & r > a \end{cases}$$

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$$\vec{J}_m = \nabla \times \vec{M} = \frac{\mu - \mu_o}{\mu_o} J_o \vec{i}_z$$

$$K_{mz} = -M_\phi(r=a_-) = -\left(\frac{\mu - \mu_o}{\mu_o}\right) \frac{J_o a}{2}$$

$$b) \quad H_x = \begin{cases} -\frac{K_o}{2} & y > 0 \\ +\frac{K_o}{2} & y < 0 \end{cases}; \quad B_x = \begin{cases} -\frac{\mu K_o}{2} & 0 < y < \frac{d}{2} \\ \frac{\mu K_o}{2} & -\frac{d}{2} < y < 0 \\ -\frac{\mu_o K_o}{2} & y > \frac{d}{2} \\ \frac{\mu_o K_o}{2} & y < -\frac{d}{2} \end{cases}; \quad M_x = \begin{cases} -\frac{(\mu - \mu_o) K_o}{2\mu_o} & 0 < y < \frac{d}{2} \\ \frac{(\mu - \mu_o) K_o}{2\mu_o} & -\frac{d}{2} < y < 0 \\ 0 & |y| > \frac{d}{2} \end{cases}$$

$$K_{zm}(y=0) = M_x(y=0_+) - M_x(y=0_-) = -\frac{(\mu - \mu_o) K_o}{\mu_o}$$

$$K_{zm}(y=\frac{d}{2}) = M_x(y=\frac{d}{2}_+) - M_x(y=\frac{d}{2}_-) = \frac{\mu - \mu_o}{2\mu_o} K_o$$

$$K_{zm}(y=-\frac{d}{2}) = M_x(y=-\frac{d}{2}_+) - M_x(y=-\frac{d}{2}_-) = \frac{\mu - \mu_o}{2\mu_o} K_o$$

## Section 5.6

$$25. \quad \vec{n} \cdot [\vec{B}_2 - \vec{B}_1] = 0 \rightarrow \mu_1 H_1 \cos \theta_1 = \mu_2 H_2 \cos \theta_2$$

$$\vec{n} \times [\vec{H}_2 - \vec{H}_1] = 0 \rightarrow H_1 \sin \theta_1 = H_2 \sin \theta_2$$

$$\tan \theta_2 = \frac{\mu_2}{\mu_1} \tan \theta_1$$

$$H_2 = \frac{\mu_1}{\mu_2} H_1 \frac{\cos \theta_1}{\cos \theta_2} = \frac{\mu_1}{\mu_2} H_1 \cos \theta_1 \sqrt{1 + \tan^2 \theta_2} = \frac{\mu_1}{\mu_2} H_1 \cos \theta_1 \sqrt{1 + \left(\frac{\mu_2}{\mu_1} \tan \theta_1\right)^2}$$

$$26. \quad \text{Replace magnetization by surface current at } r = a, K_\phi = M_o.$$

a) From (5.2.4c)

$$B_z = \frac{\mu_o M_o}{2} \left\{ \frac{\frac{L}{2} - z}{[(z - \frac{L}{2})^2 + a^2]^{1/2}} + \frac{\frac{L}{2} + z}{[(z + \frac{L}{2})^2 + a^2]^{1/2}} \right\}$$

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$$H_z = \begin{cases} \frac{B_z}{\mu_0} - M_0 & |z| < \frac{L}{2} \\ \frac{B_z}{\mu_0} & |z| > \frac{L}{2} \end{cases}$$

$$\begin{aligned} \text{b) } \lim_{z \gg \frac{L}{2}} B_z &= \frac{\mu_0 M_0}{2} \left\{ \frac{-1}{\left[1 + \frac{a^2}{(z - \frac{L}{2})^2}\right]^{1/2}} + \frac{1}{\left[1 + \frac{a^2}{(z + \frac{L}{2})^2}\right]^{1/2}} \right\} \\ &= \frac{\mu_0 M_0}{2} \left\{ -1 + \frac{1}{2} \frac{a^2}{(z - \frac{L}{2})^2} + 1 - \frac{1}{2} \frac{a^2}{(z + \frac{L}{2})^2} \right\} \\ &= \frac{\mu_0 M_0 a^2}{4z^2} \left\{ \frac{1}{(1 - \frac{L}{2z})^2} - \frac{1}{(1 + \frac{L}{2z})^2} \right\} \\ &= \frac{\mu_0 M_0 a^2}{4z^2} \left\{ 1 + \frac{L}{z} - (1 - \frac{L}{z}) \right\} \\ &= \frac{\mu_0 M_0 a^2 L}{2z^3} \end{aligned}$$

Far field of dipole along z axis

$$\lim_{z \gg \frac{L}{2}} B_z = \frac{\mu_0 m}{2\pi z^3} \rightarrow m = M_0 \pi a^2 L$$

$$\text{c) } \lim_{a \rightarrow \infty} B_z = 0, \quad H_z = \begin{cases} -M_0 & |z| < \frac{L}{2} \\ 0 & |z| > \frac{L}{2} \end{cases}$$

$$\text{d) } \vec{J}_m = \nabla \times \vec{M} = -\frac{\partial M_z}{\partial r} \vec{i}_\phi = \frac{M_0}{a} \vec{i}_\phi$$

$$dK_\phi = \frac{M_0}{a} dr'$$

From (a)

$$dB_z = \frac{\mu_0 M_0}{2a} dr' \left\{ \frac{\frac{L}{2} - z}{\left[(z - \frac{L}{2})^2 + r'^2\right]^{1/2}} + \frac{\frac{L}{2} + z}{\left[(z + \frac{L}{2})^2 + r'^2\right]^{1/2}} \right\}$$

$$B_z = \int_{r=0}^a dB_z = \frac{\mu_0 M_0}{2a} \left\{ \left(\frac{L}{2} - z\right) \ln[r' + \sqrt{r'^2 + (z - \frac{L}{2})^2}] + \left(\frac{L}{2} + z\right) \ln[r' + \sqrt{r'^2 + (z + \frac{L}{2})^2}] \right\} \Big|_{r=0}^a$$

$$= \frac{\mu_0 M_0}{2a} \left\{ \left(\frac{L}{2} - z\right) \ln \frac{a + \sqrt{a^2 + (z - \frac{L}{2})^2}}{|z - \frac{L}{2}|} + \left(\frac{L}{2} + z\right) \ln \frac{a + \sqrt{a^2 + (z + \frac{L}{2})^2}}{|z + \frac{L}{2}|} \right\}$$

### Section 5.7

$$27. \quad a) \quad \bar{H}_1 = \frac{I}{2\pi} \frac{[-y\bar{i}_x + (x-d)\bar{i}_y]}{[(x-d)^2 + y^2]} + \frac{I'}{2\pi} \frac{[-y\bar{i}_x + (x+d)\bar{i}_y]}{[(x+d)^2 + y^2]}$$

$$\bar{H}_2 = \frac{I''}{2\pi} \frac{[-y\bar{i}_x + (x-d)\bar{i}_y]}{[(x-d)^2 + y^2]}$$

Boundary Conditions:

$$\text{At } x = 0: \quad \mu_1 H_{1x} = \mu_2 H_{2x} \rightarrow \mu_1 (I + I') = \mu_2 I'' \quad I' = \frac{(\mu_2 - \mu_1)I}{\mu_1 + \mu_2}$$

$\rightarrow$

$$H_{1y} = H_{2y} \quad -I + I' = -I'' \quad I'' = \frac{2\mu_1 I}{\mu_1 + \mu_2}$$

$$b) \quad \bar{f} = -\frac{\mu_1 I I'}{2\pi(2d)} \bar{i}_x = -\frac{\mu_1(\mu_2 - \mu_1)I^2}{(\mu_1 + \mu_2)4\pi d} \bar{i}_x$$

28. a) Try image  $-I$  at  $b = a^2/D$  from center and current  $I_0 + I$  at center.

$$\bar{H} = \frac{I}{2\pi} \frac{[-y\bar{i}_x + (x+D)\bar{i}_y]}{[(x+D)^2 + y^2]} - \frac{I}{2\pi} \frac{[-y\bar{i}_x + (x + \frac{a^2}{D})\bar{i}_y]}{[y^2 + (x + \frac{a^2}{D})^2]} + \frac{(I_0 + I)[-y\bar{i}_x + x\bar{i}_y]}{2\pi[x^2 + y^2]}$$

$$x\bar{i}_y - y\bar{i}_x = r\bar{i}_\phi; \quad \bar{i}_y = \sin\phi\bar{i}_r + \cos\phi\bar{i}_\phi$$

$$\bar{H} = \frac{I}{2\pi} \frac{[r\bar{i}_\phi + D\bar{i}_y]}{[(x+D)^2 + y^2]} - \frac{I}{2\pi} \frac{[r\bar{i}_\phi + \frac{a^2}{D}\bar{i}_y]}{[y^2 + (x + \frac{a^2}{D})^2]} + \frac{(I_0 + I)}{2\pi r} \bar{i}_\phi$$

$$\text{Check: } H_r(x^2 + y^2 = a^2) = 0$$

$$H_r = \frac{I}{2\pi} D \sin\phi \left[ \frac{1}{[(x+D)^2 + y^2]} - \frac{a^2/D^2}{[y^2 + (x + a^2/D)^2]} \right]$$

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$$= \frac{ID}{2\pi} \sin \phi \left[ \frac{1}{x^2 + y^2 + 2Dx + D^2} - \frac{1}{\frac{D^2}{a^2} [x^2 + y^2 + \frac{2xa^2}{D} + (\frac{a^2}{D})^2]} \right]$$

$$H_r (x^2 + y^2 = a^2) = 0$$

$$\begin{aligned} \text{a) and b) } H_\phi &= \frac{Ir}{2\pi} \left| \frac{1}{[(x+D)^2 + y^2]} - \frac{1}{[y^2 + (x + \frac{a^2}{D})^2]} \right| \\ &+ \frac{ID}{2\pi} \cos \phi \left\{ \frac{1}{[(x+D)^2 + y^2]} - \frac{a^2/D^2}{[y^2 + (x + \frac{a^2}{D})^2]} \right\} + \frac{I_o + I}{2\pi r} \end{aligned}$$

$$H_\phi (x^2 + y^2 = a^2) = \frac{Ia}{2\pi} \left\{ \frac{1}{a^2 + D^2 + 2xD} - \frac{1}{a^2 + (\frac{a^2}{D})^2 + \frac{2xa^2}{D}} \right\} + \frac{(I_o + I)}{2\pi a}$$

$$x = a \cos \phi$$

$$K_z(\phi) = H_\phi (x^2 + y^2 = a^2) = \frac{Ia}{2\pi} \left\{ \frac{1}{a^2 + D^2 + 2aD \cos \phi} - \frac{1}{[a^2 + D^2 + 2aD \cos \phi] \frac{a^2}{D^2}} \right\} + \frac{(I_o + I)}{2\pi a}$$

$$\begin{aligned} I_z &= 2 \int_{\phi=0}^{\pi} K_z a d\phi \\ &= \frac{Ia^2}{\pi} \int_{\phi=0}^{\pi} \frac{(1 - \frac{D^2}{a^2})}{a^2 + D^2 + 2aD \cos \phi} d\phi + (I_o + I) \\ &= \frac{Ia^2}{\pi} (1 - \frac{D^2}{a^2}) \frac{2}{[(a^2 + D^2)^2 - (2aD)^2]^{1/2}} \tan^{-1} \left\{ \frac{[(a^2 + D^2)^2 - (2aD)^2]^{1/2} \tan \frac{\phi}{2}}{a^2 + D^2 + 2aD} \right\} \Bigg|_{\phi=0}^{\pi} + (I_o + I) \\ &= \frac{Ia^2 (1 - \frac{D^2}{a^2})}{[(a^2 + D^2)^2 - (2aD)^2]^{1/2}} + (I_o + I) \\ &= \frac{I(a^2 - D^2)}{[(a^2 - D^2)^2]^{1/2}} + (I_o + I) \\ &= I_o \end{aligned}$$

Integration must be  $0 < \phi < \pi$  and not  $0 < \phi < 2\pi$  to avoid ambiguity in the principal value of inverse tangent function. The integrand is even in  $\phi$  so that the integral for  $\pi < \phi < 2\pi$  is the same as for  $0 < \phi < \pi$  so that we only

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integrate  $0 < \phi < \pi$  and multiply the result by 2.

$$\begin{aligned}
 \text{c) } f_x &= \frac{-I}{2\pi} \left[ \frac{-I}{D - \frac{a^2}{D}} + \frac{(I_o + I)}{D} \right] \\
 &= \frac{I^2 D}{2\pi(D^2 - a^2)} - \frac{I(I + I_o)}{2\pi D} = \frac{I^2}{2\pi} \left[ \frac{D}{D^2 - a^2} - \frac{1}{D} \right] - \frac{II_o}{2\pi D} \\
 &= \frac{I^2 a^2}{2\pi D(D^2 - a^2)} - \frac{II_o}{2\pi D}
 \end{aligned}$$

d) Place image current  $-I$  at  $y = -y_o$  so that cylinder image current is at position

$$y = d - \frac{a^2}{d + y_o} = y_o \quad [\text{to keep magnetic field tangential to cylinder and plane}]$$

$$y_o(d + y_o) = d(d + y_o) - a^2$$

$$y_o^2 = d^2 - a^2 \rightarrow y_o = \sqrt{d^2 - a^2}$$

$$\text{e) } f_y = \frac{\mu_o I^2}{2\pi(2y_o)} = \frac{\mu_o I^2}{4\pi\sqrt{d^2 - a^2}}$$

$$29. \text{ a) } \chi = \begin{cases} A_1 \sin a y e^{-ax} \\ A_2 \sin a y e^{+ax} \end{cases}; \quad \bar{H} = \nabla \chi = \begin{cases} A_1 a [\cos a y \bar{i}_y - \sin a y \bar{i}_x] e^{-ax} & x > 0 \\ A_2 a [\cos a y \bar{i}_y + \sin a y \bar{i}_x] e^{+ax} & x < 0 \end{cases}$$

b) Boundary Conditions:

$$H_y(x=0_+) - H_y(x=0_-) = K_o \cos a y \rightarrow a(A_1 - A_2) = K_o$$

$$H_x(x=0_+) = H_x(x=0_-) \rightarrow -A_1 a = A_2 a \rightarrow A_1 = -A_2 = \frac{K_o}{2a}$$

$$\text{c) } \bar{B} = \nabla \times \bar{A} = \bar{i}_x \frac{\partial A_z}{\partial y} - \bar{i}_y \frac{\partial A_z}{\partial x} = \begin{cases} \frac{\mu_o K_o}{2} [\cos a y \bar{i}_y - \sin a y \bar{i}_x] e^{-ax} & x > 0 \\ -\frac{\mu_o K_o}{2} [\cos a y \bar{i}_y + \sin a y \bar{i}_x] e^{+ax} & x < 0 \end{cases}$$



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$$A_z = \begin{cases} \frac{\mu_o K_o}{2a} \cos ay e^{-ax} & x > 0 \\ \frac{\mu_o K_o}{2a} \cos ay e^{+ax} & x < 0 \end{cases}$$

d)  $A_z = \text{constant}$  is equation of field lines.

30. a) Particular solution,  $\bar{A} = A_z(x) \bar{i}_z$

$$\nabla^2 \bar{A} = -\mu_o \bar{J} \rightarrow \frac{d^2 A_z}{dx^2} = -\mu_o J_o \sin ax \rightarrow A_z = \begin{cases} \frac{\mu_o J_o}{a^2} \sin ax & 0 < y < d \\ 0 & y > d \end{cases}$$

b) Homogeneous,  $\bar{A} = A_z(x, y) \bar{i}_z$

$$\nabla^2 \bar{A} = 0 \rightarrow \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0 \rightarrow A_z = \sin ax [A_1 e^{ay} + A_2 e^{-ay}]$$

$$A_{z\text{total}} = \begin{cases} \sin ax \left[ \frac{\mu_o J_o}{a^2} + A_1 e^{ay} + A_2 e^{-ay} \right] & 0 < y < d \\ \sin ax B_1 e^{-a(y-d)} & y > d \end{cases}$$

$$\bar{B} = \nabla \times \bar{A} = \bar{i}_x \frac{\partial A_z}{\partial y} - \bar{i}_y \frac{\partial A_z}{\partial x}$$

$$= \begin{cases} \bar{i}_x a \sin ax [A_1 e^{ay} - A_2 e^{-ay}] - \bar{i}_y a \cos ax \left[ \frac{\mu_o J_o}{a^2} + A_1 e^{ay} + A_2 e^{-ay} \right] & 0 < y < d \\ -B_1 a [\bar{i}_x \sin ax + \bar{i}_y \cos ax] e^{-a(y-d)} & y > d \end{cases}$$

Boundary Conditions:

$$B_y(y=0) = 0 \rightarrow A_1 + A_2 + \frac{\mu_o J_o}{a} = 0$$

$$B_x(y=d_-) = B_x(y=d_+) \rightarrow -B_1 = A_1 e^{ad} - A_2 e^{-ad}$$

$$B_y(y=d_-) = B_y(y=d_+) \rightarrow B_1 = \frac{\mu_o J_o}{a} + A_1 e^{ad} + A_2 e^{-ad}$$

$$A_1 = -\frac{\mu_o J_o}{2a^2} e^{-ad}, \quad A_2 = -\frac{\mu_o J_o}{a^2} \left(1 - \frac{e^{-ad}}{2}\right), \quad B_1 = \frac{\mu_o J_o}{2a^2} (1 - e^{-ad})^2$$

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$$c) \quad K_z(y=0) = -H_x(y=0) = -a \sin ax [A_1 - A_2] = \frac{J_0}{a} \sin ax (e^{-ad} - 1)$$

$$d) \quad F_{Ay} = \frac{1}{2} \mu_0 K_z(y=0) \times H_x(y=0) = -\frac{1}{2} \mu_0 K_z^2(y=0) = -\frac{\mu_0 J_0^2}{2a^2} \sin^2 ax (1 - e^{-ad})^2$$

$$\begin{aligned} f_{Ay} &= \int_0^{2\pi/a} F_{Ay} dx \\ &= -\frac{\mu_0 J_0^2}{2a^2} (1 - e^{-ad})^2 \int_0^{2\pi/a} \sin^2 ax dx \\ &= -\frac{\mu_0 J_0^2 \pi}{2a^3} (1 - e^{-ad})^2 \end{aligned}$$

$$\begin{aligned} \vec{F}_V &= \vec{J} \times \vec{B} \\ &= J_0 \sin ax \vec{i}_z \times [B_x \vec{i}_x + B_y \vec{i}_y] \\ &= J_0 \sin ax [B_x \vec{i}_y - B_y \vec{i}_x] \end{aligned}$$

$$\begin{aligned} f_{Vy} &= J_0 a \int_{y=0}^d \int_{x=0}^{2\pi/a} [A_1 e^{ay} - A_2 e^{-ay}] \sin^2 ax dx dy \\ &= \frac{J_0 \pi}{a} [A_1 e^{ay} + A_2 e^{-ay}] \Big|_{y=0}^d \\ &= \frac{-J_0^2 \pi \mu_0}{2a^3} [e^{a(y-d)} - e^{-a(y+d)} + 2e^{-ay}] \Big|_{y=0}^d \\ &= \frac{-\mu_0 \pi J_0^2}{2a^3} [1 - e^{-2ad} + 2e^{-ad} - e^{-ad} + e^{-ad} - 2] \\ &= \frac{-\mu_0 \pi J_0^2}{2a^3} [-1 - e^{-2ad} + 2e^{-ad}] \\ &= \frac{\mu_0 \pi J_0^2}{2a^3} (1 - e^{-ad})^2 \end{aligned}$$

$$f_{Ay} + f_{Vy} = 0$$

$$e) \quad \vec{M} = M_0 \sin ax \vec{i}_y \rightarrow \nabla \cdot \vec{M} = 0, \quad \nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0, \quad \nabla \times \vec{H} = 0 \rightarrow \vec{H} = \nabla \chi, \quad \nabla^2 \chi = 0$$

# THE MAGNETIC FIELD

$$\chi = \begin{cases} \sin ax [A_1 e^{ay} + A_2 e^{-ay}] & 0 < y < d \\ B_1 \sin ax e^{-a(y-d)} & y > d \end{cases}$$

$$\bar{H} = \nabla \chi = \begin{cases} a[\bar{i}_x \cos ax (A_1 e^{ay} + A_2 e^{-ay}) + \bar{i}_y \sin ax (A_1 e^{ay} - A_2 e^{-ay})] & 0 < y < d \\ B_1 a[\bar{i}_x \cos ax - \bar{i}_y \sin ax] e^{-a(y-d)} & y > d \end{cases}$$

Boundary Conditions:

$$B_y(y=0) = \mu_o [H_y(y=0) + M_o \sin ax] = 0 \rightarrow \frac{M_o}{a} + A_1 - A_2 = 0$$

$$H_x(y=d_+) = H_x(y=d_-) \rightarrow B_1 = A_1 e^{ad} + A_2 e^{-ad}$$

$$B_y(y=d_+) = B_y(y=d_-) \rightarrow -B_1 = A_1 e^{ad} - A_2 e^{-ad} + \frac{M_o}{a}$$

$$A_1 = -\frac{M_o}{2a} e^{-ad}, \quad A_2 = \frac{M_o}{a} \left(1 - \frac{e^{-ad}}{2}\right), \quad B_1 = -\frac{M_o}{2a} (1 - e^{-ad})^2$$

$$K_z(y=0) = -H_x(y=0) = -a \cos ax (A_1 + A_2) = -M_o \cos ax (1 - e^{-ad})$$

$$F_{Ay} = \frac{1}{2} \mu_o K_z(y=0) H_x(y=0) = -\frac{1}{2} \mu_o K_z^2(y=0) = -\frac{M_o^2}{2} \mu_o \cos^2 ax (1 - e^{-ad})^2$$

$$f_{Ay} = \int_{x=0}^{2\pi/a} F_{Ay} dx = -\frac{\mu_o M_o^2}{2} (1 - e^{-ad})^2 \int_{x=0}^{2\pi/a} \cos^2 ax dx = -\frac{\mu_o M_o^2 \pi}{2a} (1 - e^{-ad})^2$$

$$\bar{F}_V = \mu_o (\bar{M} \cdot \nabla) \bar{H} = \mu_o M_o \sin ax \frac{\partial}{\partial y} [H_x \bar{i}_x + H_y \bar{i}_y]$$

$$F_{Vy} = \mu_o M_o \sin^2 ax a^2 [A_1 e^{ay} + A_2 e^{-ay}]$$

$$f_{Vy} = \mu_o M_o a^2 \int_{y=0}^d \int_{x=0}^{2\pi/a} \sin^2 ax [A_1 e^{ay} + A_2 e^{-ay}] dx dy$$

$$= \mu_o M_o \pi [A_1 e^{ay} - A_2 e^{-ay}] \Big|_{y=0}^d$$

# THE MAGNETIC FIELD

$$\begin{aligned}
 &= \mu_o M_o \pi [A_1 (e^{ad} - 1) - A_2 (e^{-ad} - 1)] \\
 &= \frac{\mu_o M_o^2 \pi}{2a} (1 - 2e^{-ad} + e^{-2ad}) = \frac{\mu_o M_o^2 \pi}{2a} (1 - e^{-ad})^2 = -f_{Ay}
 \end{aligned}$$

31. a)

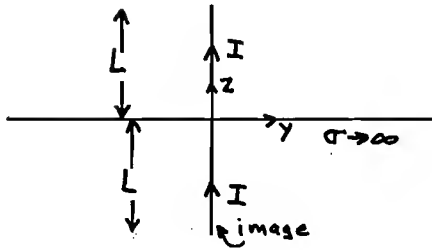


Image current in the same direction as original current.

$$b) \quad \vec{B} = \frac{\mu_o I}{4\pi r} \left\{ \frac{L - z}{[r^2 + (L - z)^2]^{1/2}} + \frac{L + z}{[r^2 + (L + z)^2]^{1/2}} \right\} \vec{i}_\phi$$

$$c) \quad \vec{K}(z=0) = -\vec{i}_z \times \vec{H}(z=0) = \frac{-I L}{2\pi r [r^2 + (L)^2]^{1/2}} \vec{i}_r$$

$$32. \quad \nabla^2 \chi = 0, \quad \vec{H} = \nabla \chi$$

$$\chi = \begin{cases} \text{Arcos} \phi & r \leq a \\ (Br + \frac{C}{r}) \cos \phi & r \geq a \end{cases}$$

$$\vec{H} = \nabla \chi = \begin{cases} A [\cos \phi \vec{i}_r - \sin \phi \vec{i}_\phi] & r < a \\ [B - \frac{C}{r^2}] \cos \phi \vec{i}_r - [B + \frac{C}{r^2}] \sin \phi \vec{i}_\phi & r > a \end{cases}$$

Boundary Conditions:

$$\vec{H}(r \rightarrow \infty) = H_o \vec{i}_x = H_o [\vec{i}_r \cos \phi - \vec{i}_\phi \sin \phi] \rightarrow B = H_o$$

$$a) \quad H_\phi(r=a_+) = H_\phi(r=a_-) \rightarrow A = B + \frac{C}{a^2}$$

$$A = \frac{2\mu_1 H_o}{\mu_1 + \mu_2}$$

$$\mu_1 H_r(r=a_+) = \mu_2 H_r(r=a_-) \rightarrow \mu_2 A = \mu_1 (B - \frac{C}{a^2})$$

$$C = \frac{H_o a^2 (\mu_1 - \mu_2)}{\mu_1 + \mu_2}$$

# THE MAGNETIC FIELD

$$b) \quad A = 0, \quad H_r(r=a_+) = 0 \quad B = \frac{C}{2} = H_0$$

$$\bar{H} = \begin{cases} 0 & r < a \\ H_0 \left\{ \left[ 1 - \frac{a^2}{r^2} \right] \cos \phi \bar{i}_r - \left[ 1 + \frac{a^2}{r^2} \right] \sin \phi \bar{i}_\phi \right\} & r > a \end{cases}$$

$$c) \quad H_\phi(r=a_+) = H_\phi(r=a_-) \rightarrow A = B + \frac{C}{2} \quad A = H_0 + \frac{(M_1 - M_2)}{2}$$

$$H_r(r=a_+) + M_1 \cos \phi = H_r(r=a_-) + M_2 \cos \phi \rightarrow B - \frac{C}{2} + M_1 = A + M_2 \quad C = a^2 \frac{(M_1 - M_2)}{2}$$

$$\bar{H} = \begin{cases} \left[ H_0 + \frac{M_1 - M_2}{2} \right] [\bar{i}_r \cos \phi - \bar{i}_\phi \sin \phi] = \left[ H_0 + \frac{M_1 - M_2}{2} \right] \bar{i}_x & r < a \\ \left[ H_0 - \frac{(M_1 - M_2)a^2}{2r^2} \right] \cos \phi \bar{i}_r - \left[ H_0 + \frac{(M_1 - M_2)a^2}{2r^2} \right] \sin \phi \bar{i}_\phi & r > a \end{cases}$$

$$33. \quad a) \quad K(y) = \frac{4K_0}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n}$$

$$b) \quad \chi = \begin{cases} A_1 \cos k y e^{-kx} & x > 0 \\ A_2 \cos k y e^{+kx} & x < 0 \end{cases}$$

$$\bar{H} = \nabla \chi = \begin{cases} A_1 k [-\sin k y \bar{i}_y - \cos k y \bar{i}_x] e^{-kx} & x > 0 \\ A_2 k [-\sin k y \bar{i}_y + \cos k y \bar{i}_x] e^{+kx} & x < 0 \end{cases}$$

$$H_x(x=0_+) = H_x(x=0_-) \rightarrow A_1 = -A_2$$

$$H_y(x=0_+) - H_y(x=0_-) = K_n \sin \frac{n\pi y}{d} \rightarrow \frac{n\pi}{d} [-A_1 + A_2] = K_n; \quad k_n = \frac{n\pi}{d}$$

$$A_1 = -A_2 = \frac{-K_n d}{2n\pi}; \quad K_n = \frac{4K_0}{n\pi}$$

# THE MAGNETIC FIELD

$$c) \quad \vec{H} = \begin{cases} \sum_{n \text{ odd}} \frac{2K_o}{n\pi} \left[ \sin \frac{n\pi y}{d} \vec{i}_y + \cos \frac{n\pi y}{d} \vec{i}_x \right] e^{-n\pi x/d} & x > 0 \\ \sum_{n \text{ odd}} \frac{2K_o}{n\pi} \left[ -\sin \frac{n\pi y}{d} \vec{i}_y + \cos \frac{n\pi y}{d} \vec{i}_x \right] e^{+n\pi x/d} & x < 0 \end{cases}$$

$$d) \quad K_z(y=0) = -H_x(y=0) = \begin{cases} \sum_{n \text{ odd}} \frac{-2K_o}{n\pi} e^{-n\pi x/d} & x > 0 \\ \sum_{n \text{ odd}} \frac{-2K_o}{n\pi} e^{+n\pi x/d} & x < 0 \end{cases}$$

$$K_z(y=d) = H_x(y=d) = K_z(y=0)$$

On  $y = 0$  for  $x > 0$

$$\begin{aligned} I_z &= \int_{x=0}^{\infty} K_z(x) dx \\ &= \frac{-2K_o}{\pi} \int_0^{\infty} \sum_{n \text{ odd}} \frac{1}{n} e^{-n\pi x/d} dx \\ &= \frac{-2K_o}{\pi} \sum_{n \text{ odd}} \frac{d}{n^2 \pi} e^{-n\pi x/d} \Big|_{x=0}^{\infty} \\ &= \frac{2K_o d}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{K_o d}{4} \end{aligned}$$

$$I_z(x>0, y=0) = I_z(x<0, y=0) = I_z(x>0, y=d) = I_z(x<0, y=d) = \frac{K_o d}{4}$$

## Section 5.8

34. a) From Prob. (32c) with  $M_1 = 0$ ,  $M_2 = M_o$ ,  $H_o = 0$

$$\vec{H} = \begin{cases} -\frac{M_o}{2} [\vec{i}_r \cos \phi - \vec{i}_\phi \sin \phi] = -\frac{M_o}{2} \vec{i}_x & r < a \\ \frac{M_o a^2}{2r^2} [\cos \phi \vec{i}_r + \sin \phi \vec{i}_\phi] & r > a \end{cases}$$

$$b) \quad \vec{f}_L = -\vec{i}_z \times \mu_o \vec{H}(\phi = -90^\circ) \quad (I \text{ at } y = -b)$$

$$= \frac{\mu_o I M_o a^2}{2b^2} (\vec{i}_z \times \vec{i}_\phi)$$

# THE MAGNETIC FIELD

$$= \frac{\mu_o I M_o a^2}{2b^2} \bar{i}_y$$

$$\bar{f}_L = -\bar{i}_z \times \mu_o \bar{H}(\phi = 0^\circ)$$

$$= \frac{-\mu_o I M_o a^2}{2b^2} (\bar{i}_z \times \bar{i}_r) = \frac{-\mu_o I M_o a^2}{2b^2} \bar{i}_y$$

$$35. \quad a) \quad J_y = \frac{I}{Dd} \rightarrow \nabla \times \bar{H} = -\frac{dH_z}{dx} \bar{i}_y = J_y \bar{i}_y \rightarrow H_z(x) = \frac{-I}{Dd} x + \text{constant}$$

$$H_z(x=0) = \frac{I}{D}, \quad H_z(x=d) = 0 \rightarrow H_z(x) = \frac{-I}{Dd} (x - d)$$

$$b) \quad \bar{F} = \mu_o (\bar{M} \cdot \nabla) \bar{H} + \mu_o \bar{J}_f \times \bar{H}$$

$$= \mu_o M_z \frac{dH_z}{dz} + \frac{\mu_o I}{Dd} \bar{i}_y \times [H_z(x) \bar{i}_z]$$

$$F_x = -\frac{\mu_o I^2}{(Dd)^2} (x - d)$$

$$f_x = \int_{x=0}^d F_x s D dx$$

$$= -\frac{\mu_o I^2 s}{Dd^2} \int_{x=0}^d (x - d) dx$$

$$= -\frac{\mu_o I^2 s}{Dd^2} \frac{(x - d)^2}{2} \Big|_{x=0}^d$$

$$= \frac{1}{2} \frac{\mu_o I^2 s}{D} \quad [\text{independent of } \mu]$$

$$36. \quad \bar{F} = \mu_o (\bar{M} \cdot \nabla) \bar{H}$$

$$= (\mu - \mu_o) (\bar{H} \cdot \nabla) \bar{H}$$

$$= \nabla \left( \frac{1}{2} (\mu - \mu_o) |\bar{H}|^2 \right)$$

$$a) \quad \bar{F} = \frac{1}{2} (\mu - \mu_o) \left\{ \frac{\partial}{\partial x} [H_x^2 + H_y^2] \bar{i}_x + \frac{\partial}{\partial y} [H_x^2 + H_y^2] \bar{i}_y \right\}$$

# THE MAGNETIC FIELD

$$f_x = \int_{x=-\infty}^{x_0} \int_{y=0}^s F_x D dx dy$$

$$= \frac{1}{2} (\mu - \mu_0) D \int_{y=0}^s [H_x^2 + H_y^2] \Big|_{x=-\infty}^{x_0} dy$$

$$= \frac{1}{2} (\mu - \mu_0) D \int_{y=0}^s H_0^2 dy = \frac{1}{2} (\mu - \mu_0) H_0^2 D s$$

$$b) F_x = \mu_0 M_y \frac{\partial H}{\partial y}$$

$$f_x = \mu_0 M_y D \int_{y=0}^s H_x \Big|_{x=-\infty}^0 dy$$

$$= \mu_0 M_y D [H_0 - H_x(-\infty)] s$$

$$= \mu_0 M_0 D [H_0 + M_0] s$$



CHAPTER 6  
ELECTROMAGNETIC INDUCTION

Section 6.1

1.

$$a) \quad H_{\phi} = \frac{I}{2\pi r} = \frac{I}{2\pi [D + r \cos \phi]}$$

$$\begin{aligned} \Phi &= \int_{r=0}^a \int_{\phi=0}^{2\pi} \mu_o H_{\phi} r dr d\phi \\ &= \frac{\mu_o I}{2\pi} \int_{r=0}^a \int_{\phi=0}^{2\pi} \frac{r dr d\phi}{[D + r \cos \phi]} \\ &= \frac{\mu_o I}{\pi} \int_{r=0}^a \frac{2r}{\sqrt{D^2 - r^2}} \tan^{-1} \left\{ \frac{\sqrt{D^2 - r^2} \tan \frac{\phi}{2}}{D + r} \right\} \bigg|_{\phi=0}^{\pi} dr \\ &= \mu_o I \int_{r=0}^a \frac{r}{\sqrt{D^2 - r^2}} dr \\ &= -\mu_o I \left[ \sqrt{D^2 - r^2} \right]_{r=0}^a \\ &= -\mu_o I \left[ \sqrt{D^2 - a^2} - D \right] \end{aligned}$$

$$M = \frac{\Phi}{I} = \mu_o \left[ D - \sqrt{D^2 - a^2} \right]$$

$$R = \frac{2\pi a}{\sigma A}$$

$$b) \quad -iR = M \frac{di}{dt} + L \frac{di}{dt} \rightarrow L \frac{di}{dt} + iR = M \frac{dI}{dt} \rightarrow i = \frac{MI}{L} e^{-t/\tau} ; \tau = \frac{L}{R}$$

$$c) \quad i(t) = -\frac{MI}{L} e^{-(t - T)/\tau}$$

$$d) \quad R = 0 \rightarrow \text{Short circuit: } \frac{d}{dt} [M(D)I - Li] = 0 \rightarrow M(D)I - Li = M(D_o)I$$

$$M(D) = \mu_o \left[ D - \sqrt{D^2 - a^2} \right] ; D = D_o + v_r t$$

# ELECTROMAGNETIC INDUCTION

$$\text{Open circuit: } v_{oc} = \frac{d\Phi}{dt} = -I \frac{dM}{dt} = -I \frac{dM}{dD} \frac{dD}{dt} = -\mu_o I \left[ 1 - \frac{D}{\sqrt{D^2 - a^2}} \right] v_r$$

$$e) \quad df_r = - \frac{I a d\phi \mu_o I \cos \phi}{2\pi [D + a \cos \phi]}$$

$$\begin{aligned} f_r &= - \frac{\mu_o I i a}{2\pi} \int_{\phi=0}^{\pi} \frac{2 \cos \phi d\phi}{[D + a \cos \phi]} \\ &= - \frac{\mu_o I i a}{\pi} \left\{ -\frac{1}{a} \sin^{-1}[\cos \phi] + \frac{D}{a \sqrt{D^2 - a^2}} \sin^{-1} \left[ \frac{a + D \cos \phi}{D + a \cos \phi} \right] \right\} \Big|_{\phi=0}^{\pi} \\ &= \frac{\mu_o I i}{\pi} \left\{ -\frac{\pi}{2} - \frac{\pi}{2} - \frac{D}{\sqrt{D^2 - a^2}} \left[ -\frac{\pi}{2} - \frac{\pi}{2} \right] \right\} \\ &= \mu_o I i \left\{ \frac{D}{\sqrt{D^2 - a^2}} - 1 \right\} \end{aligned}$$

2.

$$a) \quad H_z = \begin{cases} -\frac{K_o}{2} & x > 0 \\ +\frac{K_o}{2} & x < 0 \end{cases}$$

$$\Phi = \begin{cases} -\frac{\mu_o K_o a b}{2} & t \leq 0 \\ +\frac{\mu_o K_o a b}{2} & t \geq \frac{b}{U} \\ -\frac{\mu_o K_o a (b - 2Ut)}{2} & 0 \leq t \leq \frac{b}{U} \end{cases}$$

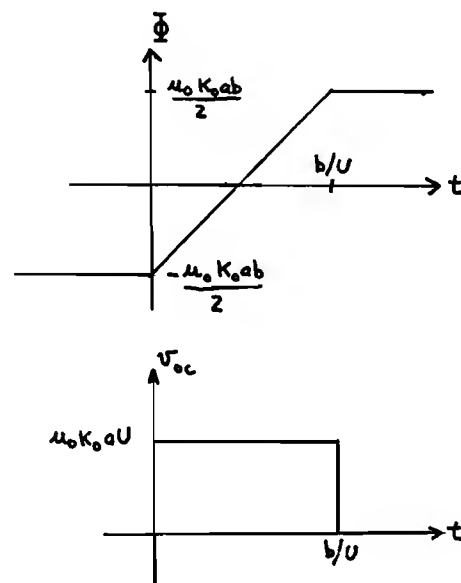
$$v_{oc} = \frac{d\Phi}{dt} = \begin{cases} 0 & t < 0, t > \frac{b}{U} \\ \mu_o K_o a U & 0 < t < \frac{b}{U} \end{cases}$$

$$b) \quad L \frac{di}{dt} + iR = - \frac{d\Phi}{dt}$$

$$t < 0, i = 0$$

$$0 < t < \frac{b}{U}; L \frac{di}{dt} + iR = -\mu_o K_o a U \rightarrow i = -\frac{\mu_o K_o a U}{R} \left( 1 - e^{-t/\tau} \right); \tau = \frac{L}{R}$$

$$t > \frac{b}{U} \quad L \frac{di}{dt} + iR = 0 \rightarrow i = -\frac{\mu_o K_o a U}{R} \left( 1 - e^{-\frac{b}{U\tau}} \right) e^{-t/\tau}$$

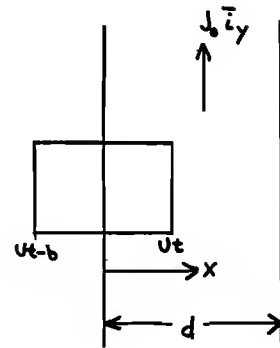


# ELECTROMAGNETIC INDUCTION

$$c) \quad H_z = \begin{cases} -\frac{J_o d}{2} & x > d \\ \frac{J_o d}{2} & x < 0 \\ \frac{J_o d}{2} \left(1 - \frac{2x}{d}\right) & 0 < x < d \end{cases} \quad \Phi = \begin{cases} -\frac{\mu_o J_o a b d}{2} & t < 0 \\ \frac{\mu_o J_o a b d}{2} & t > \frac{d+b}{U} \end{cases}$$

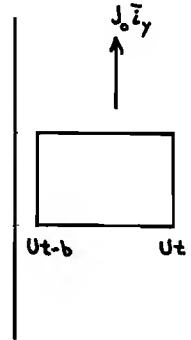
$$d > b, \quad 0 < t < b/U$$

$$\begin{aligned} \Phi &= -\frac{\mu_o J_o d a (b - Ut)}{2} - \frac{\mu_o J_o d a}{2} \int_{x=0}^{Ut} \left(1 - \frac{2x}{d}\right) dx \\ &= -\frac{\mu_o J_o d a}{2} \left[ b - Ut - \frac{d \left(1 - \frac{2x}{d}\right)^2}{4} \right]_{x=0}^{Ut} \\ &= -\frac{\mu_o J_o a d}{2} \left[ b - \frac{(Ut)^2}{d} \right] \end{aligned}$$



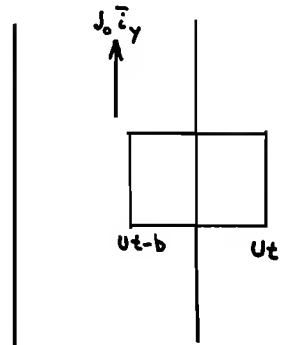
$$\frac{b}{U} < t < \frac{d}{U}$$

$$\begin{aligned} \Phi &= -\frac{\mu_o J_o d a}{2} \int_{Ut-b}^{Ut} \left(1 - \frac{2x}{d}\right) dx \\ &= \frac{\mu_o J_o d^2 a}{8} \left(1 - \frac{2x}{d}\right)^2 \Big|_{Ut-b}^{Ut} \\ &= -\frac{\mu_o J_o d a b}{2} \left[ \frac{b}{d} + 1 - \frac{2Ut}{d} \right] \end{aligned}$$



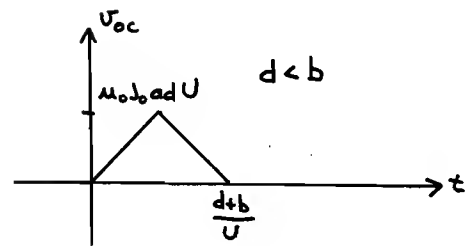
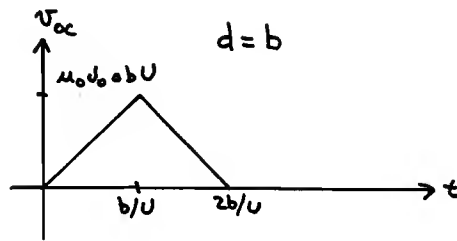
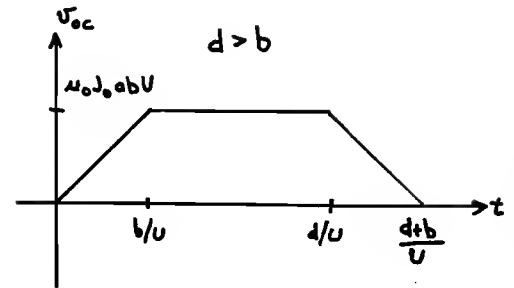
$$\frac{d}{U} < t < \frac{d+b}{U}$$

$$\begin{aligned} \Phi &= \frac{\mu_o J_o a d}{2} (Ut - d) - \frac{\mu_o J_o a d}{2} \int_{Ut-b}^d \left(1 - \frac{2x}{d}\right) dx \\ &= \frac{\mu_o J_o a d}{2} \left[ Ut - d + \frac{d}{4} \left(1 - \frac{2x}{d}\right)^2 \right]_{x=Ut-b}^d \\ &= \frac{\mu_o J_o a d}{2} \left[ 2Ut - (d+b) - \frac{(Ut-b)^2}{d} \right] \end{aligned}$$



# ELECTROMAGNETIC INDUCTION

$$v_{oc} = \frac{d\Phi}{dt} = \begin{cases} 0 & t \leq 0 \\ \mu_o J_o a U^2 t & 0 \leq t \leq \frac{b}{U} \\ \mu_o J_o a b U & \frac{b}{U} \leq t \leq \frac{d}{U} \\ \mu_o J_o a b U \left[ \frac{d}{b} - \frac{(Ut - b)}{b} \right] & \frac{d}{U} \leq t \leq \frac{d+b}{U} \\ 0 & t \geq \frac{d+b}{U} \end{cases}$$



3.

a)  $\Phi = B_o \times b \quad 0 < x < s$

b)  $\frac{L di}{dt} + iR = \frac{d\Phi}{dt} = B_o b \frac{dx}{dt} = B_o b v ; R = \frac{2(a+b)}{\sigma A}$

c)  $m \frac{dv}{dt} = f_x = -i B_o b \rightarrow i = -\frac{m}{B_o b} \frac{dv}{dt}$

$$-\frac{mL}{B_o b} \frac{d^2 v}{dt^2} - \frac{mR}{B_o b} \frac{dv}{dt} = B_o b v$$

Let  $\omega_o^2 = \frac{B_o^2 b^2}{mL}, \alpha = \frac{R}{L} \rightarrow \frac{d^2 v}{dt^2} + \alpha \frac{dv}{dt} + \omega_o^2 v = 0$

d)  $v = V e^{st} \rightarrow s^2 + \alpha s + \omega_o^2 = 0 \rightarrow s = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \omega_o^2}$

$$v(t) = \left[ V_1 \sin \beta t + V_2 \cos \beta t \right] e^{-\frac{\alpha}{2} t} ; \beta = \sqrt{\omega_o^2 - \left(\frac{\alpha}{2}\right)^2}$$

$$i(t) = -\frac{m}{B_o b} \frac{dv}{dt} = -\frac{m}{B_o b} \left[ \left( -\frac{\alpha}{2} V_1 - \beta V_2 \right) \sin \beta t + \left( -\frac{\alpha}{2} V_2 + \beta V_1 \right) \cos \beta t \right] e^{-\frac{\alpha}{2} t}$$

$$v(t=0) = v_o = V_2$$

$$i(t=0) = 0 \rightarrow \beta V_1 = \frac{\alpha}{2} V_2 \rightarrow V_1 = \frac{\alpha}{2\beta} v_o, V_2 = v_o$$

# ELECTROMAGNETIC INDUCTION

$$v(t) = v_o \left[ \frac{\alpha}{2\beta} \sin \beta t + \cos \beta t \right] e^{-\frac{\alpha}{2} t}$$

$$i(t) = \frac{m}{B_o b} v_o \left[ \frac{\alpha^2}{4\beta} + \beta \right] \sin \beta t e^{-\frac{\alpha}{2} t} = \frac{mv_o}{B_o b \beta} \omega_o^2 \sin \beta t e^{-\frac{\alpha}{2} t}$$

e)  $\sigma \rightarrow \infty \rightarrow \alpha \rightarrow 0, \beta = \omega_o$

$$v(t) = v_o \cos \omega_o t$$

$$i(t) = \frac{mv_o \omega_o}{B_o b} \sin \omega_o t$$

$$x(t) = \int v(t) dt = \frac{v_o}{\omega_o} \sin \omega_o t$$

$$x_{\max} > s \rightarrow \frac{v_o}{\omega_o} > s \rightarrow v_o > \frac{B_o b s}{\sqrt{mL}}$$

4.

a)  $H_\phi = \frac{I}{2\pi r}$

Rectangular cross section toroid

$$\Phi = \mu_o N s \int_{r=b}^a H_\phi dr$$

$$= \frac{\mu_o N s I}{2\pi} \ln \frac{a}{b} \rightarrow M = \frac{\Phi}{I} = \frac{\mu_o N s}{2\pi} \ln \frac{a}{b}$$

Circular cross section toroid

$$\Phi = \frac{\mu_o N I}{2\pi} \int_{r=0}^a \int_{\phi=0}^2 \frac{2r dr d\phi}{R + r \cos \phi}$$

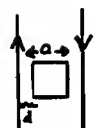
$$= \frac{\mu_o N I}{2\pi} \int_{r=0}^a r dr \frac{4}{\sqrt{R^2 - r^2}} \tan^{-1} \left\{ \frac{\sqrt{R^2 - r^2} \tan \frac{\phi}{2}}{R + r} \right\} \Big|_{\phi=0}^{\pi}$$

$$= \mu_o N I \int_{r=0}^a \frac{r}{\sqrt{R^2 - r^2}} dr$$

$$= -\mu_o N I \sqrt{R^2 - r^2} \Big|_{r=0}^a$$

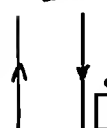
$$= \mu_o N I \left[ R - \sqrt{R^2 - a^2} \right] \rightarrow M = \frac{\Phi}{I} = \mu_o N \left[ R - \sqrt{R^2 - a^2} \right]$$

b)  $d+a < D$



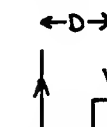
$$\Phi = \frac{\mu_0 I b}{2\pi} \left\{ \int_d^{d+a} \frac{dr}{r} + \int_{D-d-a}^{D-d} \frac{dr}{r} \right\} = \frac{\mu_0 I b}{2\pi} \ln \left[ \left( \frac{d+a}{d} \right) \left( \frac{D-d}{D-d-a} \right) \right] ; M = \frac{\Phi}{I} = \frac{\mu_0 b}{2\pi} \ln \left[ \left( \frac{d+a}{d} \right) \left( \frac{D-d}{D-d-a} \right) \right]$$

$d > D$



$$\Phi = \frac{\mu_0 I b}{2\pi} \left\{ \int_d^{d+a} \frac{dr}{r} - \int_{d-D}^{d+a-D} \frac{dr}{r} \right\} = \frac{\mu_0 I b}{2\pi} \ln \left[ \left( \frac{d+a}{d} \right) \left( \frac{d-D}{d+a-D} \right) \right] ; M = \frac{\Phi}{I} = \frac{\mu_0 b}{2\pi} \ln \left[ \left( \frac{d+a}{d} \right) \left( \frac{d-D}{d+a-D} \right) \right]$$

$d+a > D$



$$\Phi = \frac{\mu_0 I b}{2\pi} \left\{ \int_d^{d+a} \frac{dr}{r} + \int_0^{D-d} \frac{dr}{r} - \int_0^{d+a-D} \frac{dr}{r} \right\}$$

$$= \frac{\mu_0 I b}{2\pi} \ln \left[ \left( \frac{d+a}{d} \right) \left( \frac{D-d}{d+a-D} \right) \right] ; M = \frac{\Phi}{I} = \frac{\mu_0 b}{2\pi} \left\{ \ln \left[ \left( \frac{d+a}{d} \right) \left( \frac{D-d}{d+a-D} \right) \right] \right\}$$

5.

a)  $\vec{A} = \frac{\mu_0 I d S}{4\pi r^2} \sin \theta \vec{i}_\phi$ ,  $\sin \theta = \frac{a}{\sqrt{a^2 + D^2}}$ ,  $r = \sqrt{a^2 + D^2}$

b)  $A_\phi = \frac{\mu_0 m a}{4\pi (a^2 + D^2)^{3/2}} \rightarrow \Phi = \oint_L A_\phi d\phi = \frac{\mu_0 m a^2}{2 (a^2 + D^2)^{3/2}}$

c)  $M = \frac{\Phi}{I} = \frac{\mu_0 a^2 d S}{2 (a^2 + D^2)^{3/2}}$

d)  $B_z(z = -a) = \frac{\mu_0 I_2 a^2}{2 (D^2 + a^2)^{3/2}}$

e)  $\Phi = B_z(z = -a) dS = \frac{\mu_0 I_2 d S a^2}{2 (a^2 + D^2)^{3/2}}$

f)  $M = \frac{\Phi}{I_2} = \frac{\mu_0 d S a^2}{2 (a^2 + D^2)^{3/2}}$  [agrees with (c)]

6.  $H_z = \begin{cases} -\frac{K(t)}{2} & x > 0 \\ +\frac{K(t)}{2} & x < 0 \end{cases} \rightarrow \Phi = \frac{\mu_0 K_0}{2} s(b-a) = \Phi_0$

# ELECTROMAGNETIC INDUCTION

$$L \frac{di}{dt} + iR = \frac{d\Phi}{dt} ; R = \frac{2(s + a + b)}{\sigma A}$$

$$a) i(t) = \frac{\Phi_0}{L} e^{-t/\tau} ; \tau = \frac{L}{R} , (b) i(t) = -\frac{\Phi_0}{L} e^{-(t-T)/\tau}$$

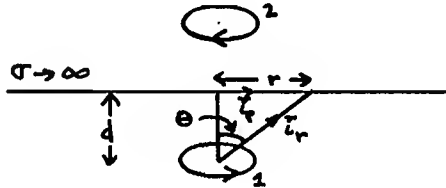
$$c) \Phi = \frac{\mu_0 K_0}{2} s(b - a) \cos \omega t = \operatorname{Re} \Phi_0 e^{j\omega t}$$

$$i(t) = \operatorname{Re} \hat{I} e^{j\omega t} \rightarrow [Lj\omega + R] \hat{I} = j\omega \Phi_0 \rightarrow \hat{I} = \frac{j\omega \Phi_0}{[Lj\omega + R]} = \frac{\Phi_0 j\omega}{[R^2 + (L\omega)^2]^{1/2}} e^{-j\phi}$$

$$\phi = \tan^{-1} \frac{L\omega}{R}$$

$$i(t) = \frac{-\omega \Phi_0}{[R^2 + (L\omega)^2]^{1/2}} \sin(\omega t - \phi)$$

7. a)



$$\bar{B}_1 = \frac{\mu_0 IdS}{4\pi r^3} [2 \cos \theta \bar{i}_r + \sin \theta \bar{i}_\theta] ; \bar{i}_r \cdot \bar{i}_\theta = \sin \theta = \frac{r}{\sqrt{r^2 + d^2}} , \bar{i}_\phi \cdot \bar{i}_r = \cos \theta = \frac{d}{\sqrt{r^2 + d^2}}$$

$$B_{r1} = \bar{B}_1 \cdot \bar{i}_r = \frac{\mu_0 IdS}{4\pi r^3} [3 \cos \theta \sin \theta]$$

$$B_r = B_{r1} + B_{r2} = \frac{3\mu_0 IdS}{2\pi (r^2 + d^2)^{3/2}} \cos \theta \sin \theta = \frac{3\mu_0 IdS dr}{2\pi (r^2 + d^2)^{5/2}}$$

$$b) K_\phi = -H_r(z = d) = -\frac{3 IdS rd}{2\pi (r^2 + d^2)^{5/2}}$$

$$c) F_z = -\frac{1}{2} \mu_0 K_\phi H_r(z = d) = \frac{1}{2} \mu_0 K_\phi^2$$

# ELECTROMAGNETIC INDUCTION

$$\begin{aligned}
 f_z &= \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} F_z \, r \, dr \, d\phi \\
 &= \frac{\mu_o 9 (IdS)^2 d^2}{4\pi} \int_{r=0}^{\infty} \frac{r^3 \, dr}{(r^2 + d^2)^5} \\
 &= - \frac{9\mu_o d^2 (IdS)^2}{4\pi} \left[ \frac{r^2 + d^2/4}{6(r^2 + d^2)^4} \right] \Big|_{r=0}^{\infty} \\
 &= \frac{3\mu_o (IdS)^2}{32\pi d^4}
 \end{aligned}$$

$$d) \quad \frac{3\mu_o (IdS)^2}{32\pi d^4} = Mg \rightarrow I^2 = \frac{32\pi Mg d^4}{3\mu_o (dS)^2} = \frac{32\pi (10^{-3}) (9.8) (10^{-8})}{3(4\pi) \times 10^{-7} (\pi 10^{-6})^2} = 2.65 \times 10^8$$

$$I = 1.63 \times 10^4 \text{ amperes}$$

8.

$$a) \quad H_z = K(t)$$

$$b) \quad \oint \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

Take contour moving with thin block

$$\int_{y=0}^s E'_y \, dy = \int_{y=0}^s \frac{K(t)}{\sigma \delta} \, dy = \frac{K(t)s}{\sigma \delta} = - \frac{d}{dt} [\mu_o s x K(t)] = -\mu_o s \left[ K(t) \frac{dx}{dt} + x \frac{dK(t)}{dt} \right]$$

$$\frac{dx}{dt} = -V \rightarrow x = x_o - Vt$$

$$\frac{dK}{dt} - \frac{KV}{x_o - Vt} \left[ 1 - \frac{1}{R_m} \right] = 0 ; R_m = \sigma \mu_o V \delta$$

$$\frac{1}{K} dK = \frac{dt}{x_o - Vt} V \left( 1 - \frac{1}{R_m} \right) \rightarrow \ln K = - \left( 1 - \frac{1}{R_m} \right) \ln (x_o - Vt)$$

$$K(x_o - Vt) \left( 1 - \frac{1}{R_m} \right) = \text{constant} = K_o x_o \left( 1 - \frac{1}{R_m} \right)$$

$$K(t) = K_o \left( \frac{x_o}{x_o - Vt} \right)^{\left( 1 - \frac{1}{R_m} \right)}$$



# ELECTROMAGNETIC INDUCTION

c)  $K(t) = K_0$  if  $R_m = \sigma \mu_0 V \delta = 1$

d) For  $R_m < 1$ ,  $K(t)$  decreases with time

For  $R_m > 1$ ,  $K(t)$  increases with time.

9. 
$$\oint \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

a)  $\Phi = \pi r^2 B(t)$ ,  $R = \frac{2\pi r}{\sigma d r d}$

$$\oint \vec{E} \cdot d\vec{l} = iR = \frac{d\Phi}{dt} = \pi r^2 \frac{dB}{dt} \rightarrow i = \frac{r\sigma d}{2} \frac{dB}{dt} dr$$

b)  $dP = i^2 R = \left( \frac{r\sigma d}{2} \frac{dB}{dt} \right)^2 dr^2 \left( \frac{2\pi r}{\sigma d r d} \right) = \frac{\pi\sigma d}{2} \left( \frac{dB}{dt} \right)^2 r^3 dr$

c)  $P = \frac{\pi\sigma d}{2} \left( \frac{dB}{dt} \right)^2 \int_{r=0}^a r^3 dr$

$$= \frac{\pi\sigma d}{2} \left( \frac{dB}{dt} \right)^2 \frac{a^4}{4} = \frac{\pi\sigma d a^4}{8} \left( \frac{dB}{dt} \right)^2$$

d)  $\pi a^2 = N\pi(a')^2 \rightarrow a' = \frac{a}{\sqrt{N}}$

e)  $P = \frac{\pi\sigma d}{8} \left( \frac{dB}{dt} \right)^2 \frac{a^4}{N^2} N = \frac{1}{N} \frac{\pi\sigma d a^4}{8} \left( \frac{dB}{dt} \right)^2$

## Section 6.2

10.  $H_\phi = \frac{Ni}{2\pi[b+rcos\theta]}$

$$\Phi = \mu_0 \int_{\theta=0}^{2\pi} \int_{r=0}^a H_\phi r dr d\theta$$

$$= \frac{\mu_0 Ni}{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a \frac{2r dr d\theta}{b+rcos\theta}$$

# ELECTROMAGNETIC INDUCTION

$$= \frac{\mu_0 N i}{\pi} \int_{r=0}^a \frac{2r dr}{\sqrt{b^2 - r^2}} \tan^{-1} \frac{\sqrt{b^2 - r^2} \tan \frac{\theta}{2}}{b + r} \bigg|_{\theta=0}^{\pi} dr$$

$$= \mu_0 N i \int_{r=0}^a \frac{r}{\sqrt{b^2 - r^2}} dr$$

$$= -\mu_0 N i \sqrt{b^2 - r^2} \bigg|_{r=0}^a = \mu_0 N i \left[ b - \sqrt{b^2 - a^2} \right]$$

$$L = \frac{N\Phi}{i} = \mu_0 N^2 \left[ b - \sqrt{b^2 - a^2} \right]$$

11. a)  $H_1 = \frac{N_1 i_1}{\ell_1}$  ,  $H_2 = \frac{N_2 i_2}{\ell_2}$

$$\Phi_2 = \mu_0 (H_1 + H_2) \pi a_2^2 = \mu_0 \pi a_2^2 \left( \frac{N_1}{\ell_1} i_1 + \frac{N_2}{\ell_2} i_2 \right)$$

$$L_2 = \frac{\mu_0 \pi a_2^2 N_2^2}{\ell_2} , L_1 = \frac{\mu_0 \pi a_1^2 N_1^2}{\ell_1} , M = \frac{\mu_0 \pi a_2^2 N_1 N_2}{\ell_2}$$

b)  $v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$  ,  $v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$

c)  $i_1 = i_2 \equiv i$

$$v = v_1 + v_2 = (L_1 + L_2 + 2M) \frac{di}{dt} \rightarrow L_t = L_1 + L_2 + 2M$$

d)  $i_1 = -i_2$

$$v = v_1 - v_2 = (L_1 + L_2 - 2M) \frac{di}{dt} \rightarrow L_t = L_1 + L_2 - 2M$$

e)  $v_1 = v_2 \rightarrow (M - L_2) \frac{di_2}{dt} = (M - L_1) \frac{di_1}{dt} \rightarrow (M - L_2) i_2 = (M - L_1) i_1$

$$i = i_1 + i_2 \Rightarrow i_1 \left[ 1 + \frac{M - L_1}{M - L_2} \right] \rightarrow i_1 = \frac{M - L_2}{2M - L_1 - L_2} i$$

# ELECTROMAGNETIC INDUCTION

$$v_1 = \frac{di_1}{dt} \left[ L_1 + \frac{M(M-L_1)}{M-L_2} \right] = \frac{di}{dt} \left[ \frac{M-L_2}{2M-L_1-L_2} \right] \left[ \frac{-L_1L_2 + M^2}{M-L_2} \right]$$

$$L_t = \frac{M^2 - L_1L_2}{2M-L_1-L_2} = \frac{L_1L_2 - M^2}{L_1+L_2-2M}$$

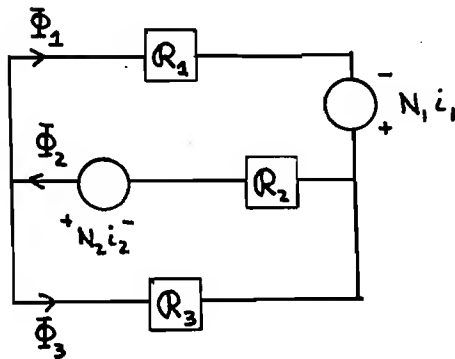
$$v_1 = -v_2 \rightarrow -(M+L_2) \frac{di_2}{dt} = (M+L_1) \frac{di_1}{dt} \rightarrow (M+L_1)i_1 = -(M+L_2)i_2$$

$$i = i_1 - i_2 = i_1 \left[ 1 + \frac{M+L_1}{M+L_2} \right] \rightarrow i_1 = \frac{(M+L_2)i}{2M+L_1+L_2}$$

$$v_1 = \frac{di_1}{dt} \left[ L_1 - \frac{M(M+L_1)}{M+L_2} \right] = \frac{di}{dt} \left[ \frac{M+L_2}{2M+L_1+L_2} \right] \left[ \frac{L_1L_2 - M^2}{M+L_2} \right]$$

$$L_t = \frac{L_1L_2 - M^2}{L_1+L_2+2M}$$

12. a)



$$R_1 = \frac{s_1}{\mu_1 a_1 D}$$

$$R_2 = \frac{s_2}{\mu_2 a_2 D}$$

$$R_3 = \frac{s_3}{\mu_3 a_3 D}$$

b)  $\Phi_2 = \Phi_1 + \Phi_3$

$$\Phi_1 R_1 + \Phi_2 R_2 = N_1 i_1 + N_2 i_2$$

$$\Phi_3 R_3 + \Phi_2 R_2 = N_2 i_2$$

$$\rightarrow \Phi_1 = \frac{N_2 i_2 R_3 + N_1 i_1 [R_2 + R_3]}{\left[ R_1 + R_3 + \frac{R_1 R_3}{R_2} \right] R_2}$$

$$\Phi_2 = \frac{N_2 i_2 [R_1 + R_3] + N_1 i_1 R_3}{R_2 \left[ R_1 + R_3 + \frac{R_1 R_3}{R_2} \right]}$$

$$\Phi_3 = \frac{N_2 i_2 R_1 - N_1 i_1 R_2}{R_2 \left[ R_1 + R_3 + \frac{R_1 R_3}{R_2} \right]}$$

# ELECTROMAGNETIC INDUCTION

$$\text{c), d) } \lambda_1 = N_1 \Phi_1 = L_1 i_1 + M i_2 ; \quad L_1 = \frac{N_1^2 [R_3 + R_2]}{R_2 \left[ R_1 + R_3 + \frac{R_1 R_3}{R_2} \right]}$$

$$\lambda_2 = N_2 \Phi_2 = M i_1 + L_2 i_2 ;$$

$$L_2 = \frac{N_2^2 [R_3 + R_1]}{R_2 \left[ R_1 + R_3 + \frac{R_1 R_3}{R_2} \right]}$$

$$M = \frac{N_1 N_2 R_3}{R_2 \left[ R_1 + R_3 + \frac{R_1 R_3}{R_2} \right]}$$

$$13. \text{ a) } H_r = \begin{cases} \frac{N_1 I_1}{2g} & 0 < \phi < \pi \\ \frac{-N_1 I_1}{2g} & \pi < \phi < 2\pi \end{cases}$$

$$\text{b) } \Phi_1 = \mu_o H_r \pi a \ell = \frac{\mu_o N_1 I_1 a \pi \ell}{2g}$$

$$L = \frac{N_1 \Phi_1}{I_1} = \frac{\mu_o N_1^2 \pi a \ell}{2g}$$

$$\text{c) } H_r = \begin{cases} \frac{[N_1 i_1 - N_2 i_2]}{2g} & 0 < \phi < \theta \\ \frac{[N_1 i_1 + N_2 i_2]}{2g} & \theta < \phi < \pi \\ \frac{[-N_1 i_1 + N_2 i_2]}{2g} & \pi < \phi < \pi + \theta \\ \frac{-[N_1 i_1 + N_2 i_2]}{2g} & \pi + \theta < \phi < 2\pi \end{cases}$$

$$\text{d) } \Phi_2 = \frac{\mu_o \ell a}{2g} \left\{ (N_1 i_1 + N_2 i_2)(\pi - \theta) + [-N_1 i_1 + N_2 i_2]\theta \right\}$$

$$= \frac{\mu_o \ell a}{2g} \left\{ N_1 i_1 (\pi - 2\theta) + N_2 i_2 \pi \right\}$$

# ELECTROMAGNETIC INDUCTION

$$L_2 = \frac{\mu_0 \ell a \pi N_2^2}{2g}, \quad M = \frac{\mu_0 \ell a}{2g} N_1 N_2 (\pi - 2\theta)$$

e)  $T = 2N_2 I_2 \ell [\mu_0 H_r] a$

$$= -\mu_0 N_1 N_2 I_1 I_2 \frac{a \ell}{2g}$$

14. a) Flux through coil 2 =  $\Phi$   $\rightarrow$   $v_2 = N_2 \frac{d\Phi}{dt}$   $\rightarrow$   $\frac{v_2}{v_1} = \frac{N_2}{2N_1}, \frac{i_2}{i_1} = \frac{2N_1}{N_2}$   
 Flux through coil 1 =  $2\Phi$   $\rightarrow$   $v_1 = 2N_1 \frac{d\Phi}{dt}$

b)  $v_2 = i_2 R_L \rightarrow v_1 \left( \frac{N_2}{2N_1} \right) = \left( \frac{2N_1}{N_2} \right) i_1 R_L$

$$\frac{v_1}{i_1} = \left( \frac{2N_1}{N_2} \right)^2 R_L$$

15. a)  $\frac{v_1}{v_2} = \frac{N}{N'}, \frac{i_1}{i_2} = \frac{N'}{N}$

b)  $v_2 = i_2 R_L \rightarrow \frac{N'}{N} v_1 = \frac{N}{N'} i_1 R_L \rightarrow \frac{v_1}{i_1} = \left( \frac{N}{N'} \right)^2 R_L$

## Section 6.3

16.  $J_{+x} = q n_+ \mu_+ E_x, \quad J_{-x} = q n_- \mu_- E_x \quad J_x = J_{+x} + J_{-x} = q (n_+ \mu_+ + n_- \mu_-) E_x$

$$J_{+y} = q \mu_+ n_+ (E_y - \mu_+ E_x B_z), \quad J_{-y} = q \mu_- n_- (E_y + \mu_- E_x B_z)$$

$$J_y = J_{+y} + J_{-y} = q E_y (\mu_+ n_+ + \mu_- n_-) - q E_x B_z (\mu_+^2 n_+ - \mu_-^2 n_-)$$

a) Open circuit  $\rightarrow J_y = 0$

$$E_y = \frac{E_x B_z (\mu_+^2 n_+ - \mu_-^2 n_-)}{\mu_+ n_+ + \mu_- n_-}$$

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$$V_{oc} = E_y d = \frac{J_x B_z d (\mu_+^2 n_+ - \mu_-^2 n_-)}{q (\mu_+ n_+ + \mu_- n_-)^2}$$

b) Short circuit  $E_y = 0$

$$i_{sc} = J_y \ell s = \frac{-J_x B_z \ell s (\mu_+^2 n_+ - \mu_-^2 n_-)}{(n_+ \mu_+ + n_- \mu_-)^2}$$

17. a)  $B_\phi = \begin{cases} \frac{\mu_o I}{2\pi r} & \text{air} \\ \frac{\mu I}{2\pi r} & \text{iron} \end{cases}$

b) Take contour moving with loop downward  $(\bar{v} = -v_o \bar{i}_y)$

$$= \oint_{EMF} \bar{E}' \cdot d\bar{\ell} + \int_{R_1}^{R_2} (\bar{v} \times \bar{B}) \cdot d\bar{r} = -\frac{d\phi}{dt} ; \quad \phi = \int_S \bar{B} \cdot d\bar{S} = \frac{-I}{2\pi} \left\{ [\mu_o y + \mu(L-y)] \int_{R_1}^{R_2} \frac{dr}{r} \right\}$$

$$= \frac{-I \ell n}{2\pi} \frac{R_2}{R_1} [(\mu_o - \mu)y + \mu L]$$

$$EMF + \int_{R_1}^{R_2} \frac{V_o \mu I}{2\pi r} dr = \frac{I}{2\pi} \ell n \frac{R_2}{R_1} (\mu - \mu_o) V_o$$

$$EMF = - \frac{\mu_o V_o I}{2\pi} \ell n \frac{R_2}{R_1}$$

c) Take stationary contour (velocity of cylinder  $\bar{v} = V_o \bar{i}_y$ )

$$\oint_{EMF} \bar{E} \cdot d\bar{\ell} - \int_{R_1}^{R_2} (\bar{v} \times \bar{B}) \cdot d\bar{r} = -\frac{d\phi}{dt} = \frac{I}{2\pi} (\mu - \mu_o) V_o \ell n \frac{R_2}{R_1}$$

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$$EMF + \frac{V_o \mu I}{2\pi} \ln \frac{R_2}{R_1} = \frac{I}{2\pi} (\mu - \mu_o) V_o \ln \frac{R_2}{R_1}$$

$$EMF = - \frac{\mu_o IV_o}{2\pi} \ln \frac{R_2}{R_1} \quad (\text{agrees with (b)})$$

d) Closed loop through slot. [Take stationary contour through loop]

$$\oint_{EMF} \vec{E} \cdot d\vec{\ell} = - \frac{d\Phi}{dt} = - \frac{IV_o}{2\pi} (\mu - \mu_o) \ln \frac{R_2}{R_1}$$

18. a)  $\vec{H} = 0, \vec{B} = \mu_o M_o \vec{i}_z$

b)  $\oint_L \vec{E} \cdot d\vec{\ell} = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} \quad (\text{Take stationary contour})$

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}, \quad \vec{J} = \sigma \vec{E}' = 0 \rightarrow \vec{E} = - \vec{v} \times \vec{B} \quad \text{in magnet}$$

$$\oint_L \vec{E} \cdot d\vec{\ell} = 0 = - v_{oc} + \int_{r=a}^b \omega r \vec{i}_\phi \times B_z \vec{i}_z \cdot d\vec{r} = -v_{oc} + \frac{\omega B_z}{2} (b^2 - a^2)$$

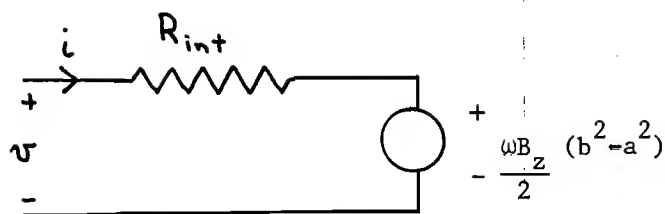
$$v_{oc} = \frac{\omega B_z}{2} (b^2 - a^2)$$

c)  $\vec{E}' = \frac{i}{2\pi r \ell \sigma} \vec{i}_r$

$$\oint \vec{E} \cdot d\vec{\ell} = 0 = -v + \int_{r=a}^b \frac{i}{2\pi r \ell \sigma} dr + \int_{r=a}^b \omega r \vec{i}_\phi \times B_z \vec{i}_z \cdot d\vec{r}$$

$$v = i R_{int} + \frac{\omega B_z}{2} (b^2 - a^2); \quad R_{int} = \frac{\ln \frac{b}{a}}{2\pi \sigma \ell}$$

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$$d) \quad v = 0 \rightarrow i = - \frac{\frac{\omega B_z}{2} (b^2 - a^2)}{2R_{int}}$$

$$J_r = \frac{i}{2\pi r \ell} = - \frac{\frac{\omega B_z}{2} (b^2 - a^2)}{4\pi r \ell R_{int}}$$

$$\vec{F} = \vec{J} \times \vec{B} = - \vec{i}_\phi \frac{\frac{\omega B_z^2}{2} (b^2 - a^2)}{4\pi r \ell R_{int}}$$

$$\begin{aligned} \vec{T} &= \int_{r=a}^b \vec{r} \times \vec{F} 2\pi r \ell dr = - \frac{\frac{\omega B_z^2}{2} (b^2 - a^2)}{2 R_{int}} \vec{i}_z \int_{r=a}^b r dr \\ &= - \frac{\frac{\omega B_z^2}{2} \left[ \frac{b^2 - a^2}{2} \right]^2}{4 R_{int}} \vec{i}_z \end{aligned}$$

$$19. \quad a) \quad \oint_L \vec{E} \cdot d\vec{\ell} = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} \quad [\text{Take contour moving with rim}]$$

$$-v + \int_a^b \frac{i}{\sigma A} dr = - \frac{d}{dt} \left\{ (B_o \cos \omega t) \frac{(2\pi - \theta)(b^2 - a^2)}{2} \right\}; \quad 0 < \theta(t) < 2\pi$$

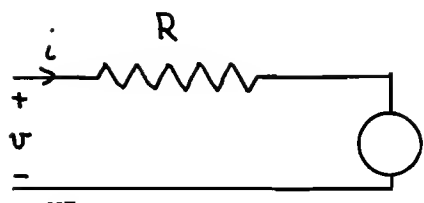
$$\begin{aligned} v &= iR - \frac{(2\pi - \theta)(b^2 - a^2)}{2} B_o \omega \sin \omega t - \frac{B_o (b^2 - a^2) \cos \omega t \omega_o}{2} \\ &= iR - \frac{B_o (b^2 - a^2)}{2} [\omega_o \cos \omega t + \omega(2\pi - \theta) \sin \omega t]; \quad R = \frac{b-a}{\sigma A} \end{aligned}$$

$$v_{oc} = v(i = 0) = - \frac{B_o (b^2 - a^2)}{2} [\omega_o \cos \omega t + \omega(2\pi - \theta) \sin \omega t]$$

$$I_{sc} = \frac{B_o (b^2 - a^2)}{2R} [\omega_o \cos \omega t + \omega(2\pi - \theta) \sin \omega t]$$



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$$\frac{B_o (b^2 - a^2)}{2} [\omega_o \cos \omega t + \omega(2\pi - \theta) \sin \omega t]$$

20. a)  $B = \frac{\mu_o N I_o}{s}$

$$J = \frac{i}{sD} = \sigma(-\frac{v}{a} + VB) ; v = -iR_{int} + VBa = iR_L$$

$$i(R_{int} + R_L) = VBa = \frac{\mu_o N I_o Va}{s} ; R_{int} = \frac{a}{\sigma s D}$$

$$i = \frac{\mu_o N I_o Va}{s(R_{int} + R_L)} , P = i^2 R_L = \left[ \frac{\mu_o N Va I_o}{s(R_{int} + R_L)} \right]^2 R_L$$

b)  $v_f = \frac{L di}{dt} , B = \frac{\mu_o N i}{s}$

$$v_f - v = \frac{L di}{dt} + i R_{int} - \frac{\mu_o N Va}{s} i = 0$$

$$i = I_o e^{-st} ; s = \frac{R_{int} - \frac{\mu_o N Va}{s}}{L}$$

For self-excitation  $s < 0 \rightarrow V > \frac{s R_{int}}{\mu_o N a}$

21.  $(L_r + L_f) \frac{di_f}{dt} + i_f (R_r + R_f - G\omega) + \frac{1}{C} \int i_f dt = 0$

$$\frac{d^2 i_f}{dt^2} + \frac{(R_r + R_f - G\omega)}{(L_r + L_f)} \frac{di_f}{dt} + \frac{1}{(L_r + L_f)C} i_f = 0$$

$$i_f = I e^{st}$$

$$s^2 + \frac{(R_r + R_f - G\omega)}{(L_r + L_f)} s + \frac{1}{(L_r + L_f)C} = 0$$

$$s = -\frac{(R_r + R_f - G\omega)}{2(L_r + L_f)} \pm \left[ \left[ \frac{(R_r + R_f - G\omega)}{2(L_r + L_f)} \right]^2 - \frac{1}{(L_r + L_f)C} \right]^{1/2}$$

a) Self excited if  $-G\omega + R_r + R_f < 0 \rightarrow \omega > \frac{(R_r + R_f)}{G}$

b) dc self-excitation

$$\left[ \frac{(R_r + R_f - G\omega)}{2(L_r + L_f)} \right]^2 - \frac{1}{(L_r + L_f)C} > 0$$

$$C > \frac{4(L_r + L_f)}{[R_r + R_f - G\omega]^2}$$

ac self-excitation

$$\left[ \frac{R_r + R_f - G\omega}{2(L_r + L_f)} \right]^2 - \frac{1}{(L_r + L_f)C} < 0 \rightarrow C < \frac{4(L_r + L_f)}{[R_r + R_f - G\omega]^2}$$

$$c) \text{ frequency } \omega_o = \left[ \frac{1}{(L_r + L_f)C} - \left[ \frac{(R_r + R_f - G\omega)}{2(L_r + L_f)} \right]^2 \right]^{1/2}$$

Section 6.4

$$22. a) H_z(t \rightarrow \infty) = 0, \quad H_z(t = 0) = \frac{I_o}{D} \left( 1 - \frac{x}{d} \right)$$

$$H_z(x = 0, t > 0) = H_z(x = d, t > 0) = 0$$

b)

$$H_z(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} e^{-\alpha_n t}; \quad \alpha_n = \frac{1}{\mu\sigma} \left( \frac{n\pi}{d} \right)^2$$

$$H_z(x, t = 0) = \frac{I_o}{D} \left( 1 - \frac{x}{d} \right) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} \rightarrow A_n = \frac{2I_o}{n\pi D} \quad [\text{See Eqs. (24)-(26)}]$$

$$H_z(x, t) = \sum_{n=1}^{\infty} \frac{2I_o}{n\pi D} \sin \frac{n\pi x}{d} e^{-\alpha_n t}$$

$$\bar{J}_f = -\bar{i}_y \frac{\partial H_z}{\partial x} = \sum_{n=1}^{\infty} -\frac{2I_o}{dD} \cos \frac{n\pi x}{d} e^{-\alpha_n t}$$

$$c) \bar{F} = \mu_o J_y H_z \bar{i}_x = -\mu_o H_z \frac{\partial H_z}{\partial x} \bar{i}_x = -\frac{\partial}{\partial x} \left( \frac{\mu_o}{2} H_z^2 \right) \bar{i}_x$$

$$f_x = sD \int_{x=0}^d F_x dx$$

$$= -s \frac{D\mu_o}{2} \left[ H_z^2(x = d) - H_z^2(x = 0) \right] = 0$$

$$23. a) H_y(t \rightarrow \infty) = H_o = \frac{NI_o}{s}$$

$$b) H_y(t = 0) = 0, \quad H_y(x = 0) = H_y(x = d) = H_o$$

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$$c) \quad H_y(x, t) = H_o + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} e^{-\alpha_n t}; \quad \alpha_n = \frac{1}{\mu\sigma} \left( \frac{n\pi}{d} \right)^2$$

$$H_y(t=0) = 0 \rightarrow \int_{x=0}^d H_o \sin \frac{n\pi x}{d} dx + \frac{A_n d}{2} = 0$$

$$- \frac{H_o d}{n\pi} \cos \frac{n\pi x}{d} \Big|_{x=0}^d + \frac{A_n d}{2} = 0 \rightarrow A_n = \frac{2H_o}{n\pi} (\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ -\frac{4H_o}{n\pi} & n \text{ odd} \end{cases}$$

$$H_y(x, t) = H_o + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{4H_o}{n\pi} \sin \frac{n\pi x}{d} e^{-\alpha_n t}$$

$$J_z = \frac{\partial H_y}{\partial x} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{4H_o}{d} \cos \frac{n\pi x}{d} e^{-\alpha_n t}$$

$$d) \quad F_x = -\mu_o J_z H_y = -\mu_o H_y \frac{\partial H_y}{\partial x} = -\frac{\partial}{\partial x} \left( \frac{1}{2} \mu_o H_y^2 \right)$$

$$f_x = sD \int_{x=0}^d F_x dx = -\frac{1}{2} \mu_o sD H_y^2 \Big|_{x=0}^d = 0$$

$$e) \quad H_y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} e^{-\alpha_n(t-T)}$$

$$H_y(t=T) = H_o = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} \rightarrow A_n = \frac{4H_o}{n\pi}$$

$$H_y = \sum_{n=1}^{\infty} \frac{4H_o}{n\pi} \sin \frac{n\pi x}{d} e^{-\alpha_n(t-T)}; \quad J_z = \frac{\partial H_y}{\partial x} = \sum_{n=1}^{\infty} \frac{4H_o}{d} \cos \frac{n\pi x}{d} e^{-\alpha_n(t-T)}$$

$$f) \quad H_y(x, t) = \operatorname{Re} \hat{H}_y(x) e^{j\omega t}$$

$$\hat{H}_y(x) = A_1 e^{(1+j)x/\delta} + A_2 e^{-(1+j)x/\delta}$$

$$H_y(x=0) = H_o = A_1 + A_2$$

$$H_y(x=d) = H_o = A_1 e^{(1+j)d/\delta} + A_2 e^{-(1+j)d/\delta}$$

$$A_1 = - \frac{H_o (1 - e^{-(1+j)d/\delta})}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}}, \quad A_2 = \frac{H_o (1 - e^{(1+j)d/\delta})}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}}$$

$$\hat{H}_y(x) = \frac{H_o}{[e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}]} \left[ -e^{(1+j)x/\delta} + e^{(1+j)(x-d)/\delta} + e^{-(1+j)x/\delta} - e^{-(1+j)(x-d)/\delta} \right]$$

$$\hat{J}_z = \frac{d\hat{H}_y}{dx} = \frac{H_o(1+j)}{\delta[e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}]} \left[ -e^{(1+j)x/\delta} + e^{(1+j)(x-d)/\delta} - e^{-(1+j)x/\delta} + e^{-(1+j)(x-d)/\delta} \right]$$

g) From (d)

$$f_x = - \frac{1}{2} \mu_o s D H_y^2 \Big|_{x=0}^d = 0$$

24. a)  $H_z(x, t) = \text{Re} \hat{H}_z(x) e^{j\omega t}$

$$\hat{H}_z(x) = H_o e^{-(1+j)x/\delta}; \quad H_o = \frac{I_o}{D}$$

$$\hat{J}_y = - \frac{d\hat{H}_z}{dx} = \frac{(1+j)}{\delta} H_o e^{-(1+j)x/\delta}$$

b)  $F_x = \mu_o J_y H_z = - \mu_o H_z \frac{\partial H_z}{\partial x} = - \frac{\partial}{\partial x} \left( \frac{1}{2} \mu_o H_z^2 \right)$

$$\begin{aligned} f_x &= s D \int_0^\infty F_x dx \\ &= - s D \int_0^\infty \frac{\partial}{\partial x} \left( \frac{1}{2} \mu_o H_z^2 \right) dx \\ &= - \frac{1}{2} \mu_o s D H_z^2 \Big|_{x=0}^\infty = \frac{1}{2} \mu_o s D H_o^2 \end{aligned}$$

c)  $\hat{H}_z(x) = A_1 e^{-(1+j)x/\delta} + A_2 e^{(1+j)x/\delta}$

$$\hat{H}_z(x=0) = H_o = A_1 + A_2$$

$$\hat{H}_z(x=d) = 0 = A_1 e^{-(1+j)d/\delta} + A_2 e^{(1+j)d/\delta}$$

$$A_1 = - \frac{H_0 e^{(1+j)d/\delta}}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}}, \quad A_2 = \frac{H_0 e^{-(1+j)d/\delta}}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}}$$

$$\hat{H}_z(x) = H_0 \left[ \frac{e^{-(1+j)(x-d)/\delta} - e^{(1+j)(x-d)/\delta}}{e^{(1+j)d/\delta} - e^{-(1+j)d/\delta}} \right]$$

$$\hat{J}_y = - \frac{d\hat{H}_z}{dx} = \frac{H_0(1+j)}{\delta} \left[ \frac{e^{-(1+j)(x-d)/\delta} + e^{(1+j)(x-d)/\delta}}{e^{(1+j)d/\delta} - e^{-(1+j)d/\delta}} \right]$$

From (b)

$$f_x = - \frac{1}{2} \mu_0 s D H_z^2 \Big|_{x=0}^d = \frac{1}{2} \mu_0 s D H_0^2 \quad (\text{unchanged})$$

$$25. \quad a) \quad H_z(t=0) = \begin{cases} -K_0 & D < y < D+d \\ 0 & 0 < y < D \end{cases} ; \quad H_z(t \rightarrow \infty) = -K_0 \quad 0 < y < D+d$$

$$J_x(y=0, t) = 0$$

$$b) \quad \frac{1}{\mu\sigma} \frac{\partial^2 \hat{H}_z}{\partial y^2} = \frac{\partial \hat{H}_z}{\partial t} ; \quad H_z(y, t) = -K_0 + \hat{H}_z(y) e^{-\alpha t}$$

$$\frac{d^2 \hat{H}_z}{dy^2} + \sigma \mu \alpha \hat{H}_z = 0 \rightarrow \hat{H}_z = A_1 \sin \sqrt{\sigma \mu \alpha} y + A_2 \cos \sqrt{\sigma \mu \alpha} y$$

$$\hat{J}_x = \frac{d\hat{H}_z}{dy} = \sqrt{\sigma \mu \alpha} \left[ A_1 \cos \sqrt{\sigma \mu \alpha} y - A_2 \sin \sqrt{\sigma \mu \alpha} y \right]$$

$$\hat{J}_x(y=0) = \sqrt{\sigma \mu \alpha} A_1 = 0 \rightarrow A_1 = 0$$

$$\hat{H}_z(y=D) = 0 = A_2 \cos \sqrt{\sigma \mu \alpha} D \rightarrow \sqrt{\sigma \mu \alpha} D = (2n+1) \frac{\pi}{2} \quad n = 0, 1, 2, \dots$$

$$\alpha = \frac{1}{\mu\sigma} \left[ \frac{(2n+1)\pi}{2D} \right]^2$$

$$H_z(y, t) = -K_o + \sum_{n=0}^{\infty} A_n \cos \frac{(2n+1)\pi y}{2D} e^{-\alpha_n t}$$

$$H_z(y, t = 0) = 0 \rightarrow K_o = \sum_{n=0}^{\infty} A_n \cos \left[ \frac{(2n+1)\pi y}{2D} \right]$$

$$\int_0^D K_o \cos \frac{(2m+1)\pi y}{2D} dy = \frac{2DK_o}{(2m+1)\pi} \sin \frac{(2m+1)\pi y}{2D} \Big|_0^D = \frac{2K_o D}{\pi (2m+1)} \sin \left[ \frac{(2m+1)\pi}{2} \right]$$

$$= \frac{(-1)^m 2K_o D}{\pi (2m+1)}$$

$$\frac{A_m D}{2} = \frac{(-1)^m 2K_o D}{\pi (2m+1)} \rightarrow A_m = \frac{4K_o (-1)^m}{\pi (2m+1)}$$

$$H_z(y, t) = -K_o + \sum_{n=0}^{\infty} \frac{4K_o (-1)^n}{\pi (2n+1)} \cos \left[ \frac{(2n+1)\pi y}{2D} \right] e^{-\alpha_n t}$$

$$J_x(y, t) = \frac{\partial H_z}{\partial y} = \sum_{n=0}^{\infty} -\frac{2K_o}{D} (-1)^n \sin \left[ \frac{(2n+1)\pi y}{2D} \right] e^{-\alpha_n t}$$

$$c) \quad H_z(y, t) = \sum_{n=0}^{\infty} A_n \cos \left[ \frac{(2n+1)\pi y}{2D} \right] e^{-\alpha_n (t-T)}$$

$$H_z(y, t=T) = -K_o = \sum_{n=0}^{\infty} A_n \cos \left[ \frac{(2n+1)\pi y}{2D} \right] \rightarrow A_n = \frac{-4K_o (-1)^n}{\pi (2n+1)}$$

$$H_z(y, t) = \sum_{n=0}^{\infty} \frac{-4K_o (-1)^n}{\pi (2n+1)} \cos \left[ \frac{(2n+1)\pi y}{2D} \right] e^{-\alpha_n (t-T)}$$

$$J_x(y, t) = \frac{\partial H_z}{\partial y} = \sum_{n=0}^{\infty} \frac{2K_o}{D} (-1)^n \sin \left[ \frac{(2n+1)\pi y}{2D} \right] e^{-\alpha_n (t-T)}$$

$$d) \quad \hat{H}_z(y) = A_1 e^{(1+j)y/\delta} + A_2 e^{-(1+j)y/\delta}$$

$$\hat{J}_x(y) = \frac{d\hat{H}_z}{dy} = \frac{(1+j)}{\delta} \left[ A_1 e^{(1+j)y/\delta} - A_2 e^{-(1+j)y/\delta} \right]$$

$$\hat{H}_z(y=D) = K_o = A_1 e^{(1+j)D/\delta} + A_2 e^{-(1+j)D/\delta}$$

$$\hat{J}_x(y=0) = 0 = \frac{(1+j)}{\delta} \left[ A_1 - A_2 \right] \rightarrow A_1 = A_2 = \left[ \frac{K_o}{e^{(1+j)D/\delta} + e^{-(1+j)D/\delta}} \right]$$

$$\hat{H}_z(y) = \frac{K_o}{[e^{(1+j)D/\delta} + e^{-(1+j)D/\delta}]} [e^{(1+j)y/\delta} + e^{-(1+j)y/\delta}]$$

$$\hat{J}_x(y=0) = \frac{K_o (1+j)}{\delta [e^{(1+j)D/\delta} + e^{-(1+j)D/\delta}]} [e^{(1+j)y/\delta} - e^{-(1+j)y/\delta}]$$

26. a)  $H_z(x) = A_1 e^{R_m x/\ell} + A_2; R_m = \sigma \mu v_o \ell$

$$H_z(x=0) = K_o = A_1 + A_2 \Rightarrow A_1 = \frac{2K_o}{1 - e^{R_m}}$$

$$H_z(x=\ell) = -K_o = A_1 e^{R_m} + A_2 \quad A_2 = \frac{-K_o [e^{R_m} + 1]}{1 - e^{R_m}}$$

$$H_z(x) = \frac{K_o}{1 - e^{R_m}} [2e^{R_m x/\ell} - (1 + e^{R_m})]$$

$$J_y(x) = \frac{-\partial H_z}{\partial x} = \frac{-K_o}{1 - e^{R_m}} \frac{2R_m}{\ell} e^{R_m x/\ell}$$

b)  $F_x = \mu_o J_y H_z = -\mu_o H_z \frac{\partial H_z}{\partial x} = -\frac{\partial}{\partial x} (\frac{1}{2} \mu_o H_z^2)$

$$f_x = sD \int_{x=0}^{\ell} F_x dx$$

$$= sD \int_{x=0}^{\ell} \frac{\partial (-\frac{1}{2} \mu_o H_z^2)}{\partial x} dx$$

$$= -\frac{1}{2} \mu_o sD H_z^2 \Big|_{x=0}^{\ell} = 0$$

27. a)  $\frac{1}{\mu_o} \nabla^2 \bar{H} = \frac{\partial \bar{H}}{\partial t} + (\bar{v} \cdot \nabla) \bar{H}; H_z(x, t) = \text{Re } \hat{H}_z(x) e^{j\omega t}$

# ELECTROMAGNETIC INDUCTION

$$\frac{1}{\mu\sigma} \frac{d^2 \hat{H}_z}{dx^2} = j\omega \hat{H}_z + v_o \frac{d\hat{H}_z}{dx} \rightarrow \frac{d^2 \hat{H}_z}{dx^2} - \mu\sigma v_o \frac{d\hat{H}_z}{dx} - j\omega\mu\sigma \hat{H}_z = 0$$

$$\hat{H}_z = A e^{px} \rightarrow p^2 - \mu\sigma v_o p - j\omega\mu\sigma = 0$$

$$p = \frac{\sigma\mu v_o}{2} \pm \sqrt{\left(\frac{\sigma\mu v_o}{2}\right)^2 + j\omega\mu\sigma}$$

$$= \frac{R_m}{2\ell} \left[ 1 \pm \sqrt{1 + j \frac{2\ell^2}{R_m^2 \delta^2}} \right] \quad ; \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$p_1 = \frac{R_m}{2\ell} + \beta$$

$$p_2 = \frac{R_m}{2\ell} - \beta \quad ; \quad \beta = \frac{R_m}{2\ell} \sqrt{1 + \frac{2j\ell^2}{R_m^2 \delta^2}}$$

$$\hat{H}_z = \left[ A_1 e^{\beta x} + A_2 e^{-\beta x} \right] e^{\frac{R_m x}{2\ell}}$$

$$\hat{H}_z(0) = K_o = A_1 + A_2$$

$$\hat{H}_z(x \rightarrow \infty) = 0 \rightarrow A_1 = 0 \rightarrow A_2 = K_o$$

$$\hat{H}_z(x) = K_o e^{-\beta x} e^{R_m x / 2\ell}$$

$$\hat{J}_y(x) = \frac{-d\hat{H}_z}{dx} = -K_o \left( \frac{R_m}{2\ell} - \beta \right) e^{-\beta x} e^{R_m x / 2\ell}$$

b)  $F_x = \mu_o J_y H_z = -\mu_o H_z \frac{\partial H_z}{\partial x} = -\frac{\partial}{\partial x} \left[ \frac{1}{2} \mu_o H_z^2 \right]$

$$f_x = sD \int_{x=0}^{\infty} F_x dx = -sD \int_{x=0}^{\infty} \frac{\partial}{\partial x} \left[ \frac{1}{2} \mu_o H_z^2 \right] dx$$

$$= -\frac{1}{2} \mu_o sD H_z^2 \Big|_0^{\infty} = \frac{1}{2} \mu_o sD K_o^2 \cos^2 \omega t$$



$$28. \quad a) \quad \hat{H}_z(x) = \begin{cases} A_1 e^{kx} + A_2 e^{-kx} & 0 < x < s \\ B e^{-\gamma(x-s)} & x > s \end{cases} \quad ; H_z(x, z, t) = \operatorname{Re} \hat{H}_z(x) e^{j(\omega t - kz)}$$

$$; \gamma^2 = k^2 (1 + jS), \quad S = \frac{\mu \sigma}{k^2} (\omega - kU)$$

Boundary Conditions:

$$\hat{H}_z(x=0) = K_0 = A_1 + A_2$$

$$\hat{H}_z(x=s_+) = \hat{H}_z(x=s_-) = A_1 e^{ks} + A_2 e^{-ks} = B$$

$$\mu_0 \hat{H}_x(x=s_-) = \mu \hat{H}_x(x=s_+)$$

$$\nabla \cdot \vec{H} = 0 \rightarrow \frac{d\hat{H}_x}{dx} - jk\hat{H}_z = 0 \rightarrow \hat{H}_x = \begin{cases} j \left[ A_1 e^{kx} - A_2 e^{-kx} \right] & 0 < x < s \\ -\frac{B}{\gamma} j k e^{-\gamma(x-s)} & x > s \end{cases}$$

$$\mu_0 \left[ A_1 e^{ks} - A_2 e^{-ks} \right] = -\frac{\mu B k}{\gamma}$$

$$B = \frac{2K_0}{\left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-ks} + \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{ks}}$$

$$A_1 = \frac{K_0 \left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-ks}}{\left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-ks} + \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{ks}}, \quad A_2 = \frac{K_0 \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{ks}}{\left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-ks} + \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{ks}}$$

$$\hat{\vec{H}}(x) = \begin{cases} \frac{K_0 \left\{ \left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{k(x-s)} (\bar{i}_z + j\bar{i}_x) + \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-k(x-s)} (\bar{i}_z - j\bar{i}_x) \right\}}{\left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-ks} + \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{ks}} & 0 < x < s \\ \frac{2K_0 e^{-\gamma(x-s)} \left[ \bar{i}_z - \frac{j k}{\gamma} \bar{i}_x \right]}{\left[ 1 - \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{-ks} + \left[ 1 + \frac{\mu}{\mu_0} \frac{k}{\gamma} \right] e^{ks}} & x > s \end{cases}$$

# ELECTROMAGNETIC INDUCTION

$$J_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \rightarrow \hat{J}_y = -jk\hat{H}_x - \frac{d\hat{H}_z}{dx}$$

$$\hat{J}_y = \begin{cases} 0 & 0 < x < s \\ \frac{2K_o \left[ \gamma^2 - k^2 \right] e^{-\gamma(x-s)}}{\gamma \left[ \left[ 1 - \frac{\mu}{\mu_o} \frac{k}{\gamma} \right] e^{-ks} + \left[ 1 + \frac{\mu}{\mu_o} \frac{k}{\gamma} \right] e^{ks} \right]} & x > s \end{cases}$$

$$\lim_{s \rightarrow 0} \begin{cases} \hat{J}_y = \frac{K_o e^{-\gamma x} (\gamma^2 - k^2)}{\gamma} \\ \hat{H} = K_o e^{-\gamma x} \left[ \hat{i}_z - \frac{jk}{\gamma} \hat{i}_x \right] \end{cases} \quad (\text{agrees with (6.4.5.)})$$

b)  $s = 0$

$$\hat{H}_z = \begin{cases} A_1 e^{\gamma x} + A_2 e^{-\gamma x} & 0 < x < d \\ B e^{-k(x-d)} & x > d \end{cases}$$

Boundary Conditions:

$$\hat{H}_z(x=0) = K_o = A_1 + A_2$$

$$\hat{H}_z(x=d_-) = \hat{H}_z(x=d_+) \rightarrow A_1 e^{\gamma d} + A_2 e^{-\gamma d} = B$$

$$\hat{H}_x = \begin{cases} \frac{jk}{\gamma} (A_1 e^{\gamma x} - A_2 e^{-\gamma x}) & 0 < x < d \\ -jB e^{-k(x-d)} & x > d \end{cases}$$

$$\mu H_x(x=d_-) = \mu_o H_x(x=d_+) \Rightarrow \frac{\mu k}{\gamma} [A_1 e^{\gamma d} - A_2 e^{-\gamma d}] = -\mu_o B$$

# ELECTROMAGNETIC INDUCTION

$$A_1 = \frac{K_o e^{-\gamma d} (1 - \frac{\mu_o \gamma}{\mu k})}{\left[ \left[ 1 - \frac{\mu_o \gamma}{\mu k} \right] e^{-\gamma d} + \left[ 1 + \frac{\mu_o \gamma}{\mu k} \right] e^{\gamma d} \right]}, \quad A_2 = \frac{K_o e^{\gamma d} (1 + \frac{\mu_o \gamma}{\mu k})}{\left[ \left[ 1 - \frac{\mu_o \gamma}{\mu k} \right] e^{-\gamma d} + \left[ 1 + \frac{\mu_o \gamma}{\mu k} \right] e^{\gamma d} \right]}$$

$$B = \frac{2K_o}{\left[ \left[ 1 - \frac{\mu_o \gamma}{\mu k} \right] e^{-\gamma d} + \left[ 1 + \frac{\mu_o \gamma}{\mu k} \right] e^{\gamma d} \right]}$$

$$\hat{H}_z = \begin{cases} \frac{K_o \left\{ \left( 1 - \frac{\mu_o \gamma}{\mu k} \right) e^{\gamma(x-d)} (\bar{i}_z + \frac{j k}{\gamma} \bar{i}_x) + \left( 1 + \frac{\mu_o \gamma}{\mu k} \right) e^{-\gamma(x-d)} (\bar{i}_z - \frac{j k}{\gamma} \bar{i}_x) \right\}}{\left[ \left[ 1 - \frac{\mu_o \gamma}{\mu k} \right] e^{-\gamma d} + \left[ 1 + \frac{\mu_o \gamma}{\mu k} \right] e^{\gamma d} \right]} & 0 < x < d \\ \frac{2K_o (\bar{i}_z - j \bar{i}_x) e^{-k(x-d)}}{\left[ \left[ 1 - \frac{\mu_o \gamma}{\mu k} \right] e^{-\gamma d} + \left[ 1 + \frac{\mu_o \gamma}{\mu k} \right] e^{\gamma d} \right]} & x > d \end{cases}$$

$$J_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \rightarrow \hat{J}_y = -jk \hat{H}_x - \frac{d \hat{H}_z}{dx}$$

$$= \begin{cases} \frac{K_o \left\{ \left( 1 - \frac{\mu_o \gamma}{\mu k} \right) e^{\gamma(x-d)} - \left( 1 + \frac{\mu_o \gamma}{\mu k} \right) e^{-\gamma(x-d)} \right\} \frac{(k^2 - \gamma^2)}{\gamma}}{\left[ \left[ 1 - \frac{\mu_o \gamma}{\mu k} \right] e^{-\gamma d} + \left[ 1 + \frac{\mu_o \gamma}{\mu k} \right] e^{\gamma d} \right]} & 0 < x < d \\ 0 & x > d \end{cases}$$

Check:

$$\hat{H}_z = K_o (\bar{i}_z - \frac{j k}{\gamma} \bar{i}_x) e^{-\gamma x}$$

$$\lim_{d \rightarrow \infty} \hat{J}_y = \frac{K_o}{\gamma} (\gamma^2 - k^2) e^{-\gamma x} \quad (\text{agrees with (6.4.5)})$$

# ELECTROMAGNETIC INDUCTION

29. a)  $\nabla^2 \bar{H} - \frac{\omega^2}{c^2} \bar{H} = 0$  ;  $c^2 = \frac{1}{\epsilon\mu}$  ;  $\bar{H} = H_y(x) \bar{i}_y$

$$\frac{d^2 H_y}{dx^2} - \frac{\omega^2}{c^2} H_y = 0 \rightarrow H_y = A_1 \sinh kx + A_2 \cosh kx ; k = \frac{\omega}{c}$$

$$H_y(x=0) = H_0 = A_2 ; H_0 = \frac{NI_0}{s} \cos \omega t$$

$$H_y(x=d) = H_0 = A_1 \sinh kd + A_2 \cosh kd \rightarrow A_1 = \frac{H_0 (1 - \cosh kd)}{\sinh kd}$$

$$H_y = H_0 \left\{ \frac{(1 - \cosh kd)}{\sinh kd} \sinh kx + \cosh kx \right\}$$

$$= H_0 \left\{ \frac{-2 \sinh^2 \frac{kd}{2} \sinh kx + \cosh kx}{2 \sinh \frac{kd}{2} \cosh \frac{kd}{2}} \right\}$$

$$= \frac{H_0}{\cosh \frac{kd}{2}} \left\{ \cosh kx \cosh \frac{kd}{2} - \sinh \frac{kd}{2} \sinh kx \right\}$$

$$= \frac{H_0}{\cosh \frac{kd}{2}} \cosh k \left( x - \frac{d}{2} \right)$$

$$J_z = \frac{\partial H_y}{\partial x} = \frac{k H_0}{\cosh \frac{kd}{2}} \sinh k \left( x - \frac{d}{2} \right)$$

b)  $F_x = -\mu_0 J_z H_y = -\mu_0 H_y \frac{\partial H_y}{\partial x} = -\frac{\partial}{\partial x} \left( \frac{1}{2} \mu_0 H_y^2 \right)$

$$f_x = sD \int_0^d F_x dx$$

$$\begin{aligned}
 &= -sD \int_0^d \frac{\partial}{\partial x} \left( \frac{1}{2} \mu_0 H_y^2 \right) dx \\
 &= -\frac{1}{2} \mu_0 sD H_y^2 \Big|_0^d = 0
 \end{aligned}$$

Section 6.5

30.

$$H_\phi = \begin{cases} 0 & 0 < r < a, \quad r > d \\ \frac{i(r - \frac{a^2}{r})}{2\pi(b^2 - a^2)} & a < r < b \\ \frac{i}{2\pi} & b < r < c \\ \frac{-i(r - \frac{d^2}{r})}{2\pi(d^2 - c^2)} & c < r < d \end{cases}$$

$$w = \frac{1}{2} \mu H_\phi^2 = \begin{cases} 0 & 0 < r < a, \quad r > d \\ \frac{1}{2} \mu_1 \left[ \frac{i}{2\pi(b^2 - a^2)} \right]^2 \left[ r^2 + \frac{a^4}{r^2} - 2a^2 \right] & a < r < b \\ \frac{1}{2} \mu_0 \left( \frac{i}{2\pi r} \right)^2 & b < r < c \\ \frac{1}{2} \mu_2 \left[ \frac{i}{2\pi(d^2 - c^2)} \right]^2 \left[ r^2 + \frac{d^4}{r^2} - 2d^2 \right] & c < r < d \end{cases}$$

$$W = 2\pi \ell \int_{r=0}^d w r dr$$

$$\begin{aligned}
&= 2\pi\ell \left\{ \frac{1}{2} \mu_1 \left[ \frac{i}{2\pi(b^2 - a^2)} \right]^2 \int_{r=a}^b \left[ r^3 + \frac{a^4}{r} - 2a^2 r \right] dr + \frac{1}{2} \mu_0 \left( \frac{i}{2\pi} \right)^2 \int_{r=b}^c \frac{dr}{r} \right. \\
&\quad \left. + \frac{1}{2} \mu_2 \left[ \frac{i}{2\pi(d^2 - c^2)} \right]^2 \int_{r=c}^d \left[ r^3 + \frac{d^4}{r} - 2d^2 r \right] dr \right\} \\
&= \frac{i^2 \ell}{4\pi} \left\{ \frac{\mu_1}{(b^2 - a^2)^2} \left[ \frac{b^4 - a^4}{4} + a^4 \ln \frac{b}{a} - a^2(b^2 - a^2) \right] + \mu_0 \ln \frac{c}{b} \right. \\
&\quad \left. + \frac{\mu_2}{(d^2 - c^2)^2} \left[ \frac{d^4 - c^4}{4} + d^4 \ln \frac{d}{c} - d^2(d^2 - c^2) \right] \right\} \\
W &= \frac{1}{2} Li^2 \Rightarrow L = \frac{2W}{i^2} = \frac{\ell}{2\pi} \left\{ \mu_1 \left[ \frac{b^2 - 3a^2}{4(b^2 - a^2)} + \frac{a^4}{(b^2 - a^2)^2} \ln \frac{b}{a} \right] + \mu_0 \ln \frac{c}{b} \right. \\
&\quad \left. + \mu_2 \left[ \frac{c^2 - 3d^2}{4(d^2 - c^2)} + \frac{d^4 \ln \frac{d}{c}}{(d^2 - c^2)^2} \right] \right\}
\end{aligned}$$

$$31. \quad L' C' = \epsilon \mu \Rightarrow L = L' \ell = \frac{\epsilon \mu \ell}{C'}$$

#### Adjacent or Enclosed Cylinders

$$C' = \frac{2\pi\epsilon}{\cosh^{-1} \left[ \frac{|D^2 - R_1^2 - R_2^2|}{2 R_1 R_2} \right]} \Rightarrow L = \frac{\mu \ell}{2\pi} \cosh^{-1} \left[ \frac{|D^2 - R_1^2 - R_2^2|}{2 R_1 R_2} \right]$$

Cylinder Plane

$$C' = \frac{2\pi\epsilon}{\cosh^{-1} \frac{s+R_2}{R_2}} \Rightarrow L = \frac{\mu\ell}{2\pi} \cosh^{-1} \left( \frac{s+R_2}{R_2} \right)$$

$$32. \quad \frac{1}{\mu\sigma} \nabla^2 \bar{H} = \frac{\partial \bar{H}}{\partial t} ; \quad \bar{H} = \text{Re } \hat{H}_\phi(r) e^{j\omega t} \bar{i}_\phi$$

$$\nabla \times (\hat{H}(\phi) \bar{i}_\phi) = \frac{1}{r} \frac{\partial}{\partial r} (r \hat{H}_\phi) \bar{i}_z$$

$$\nabla \times (\nabla \times \bar{H}) = - \frac{\partial}{\partial t} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \hat{H}_\phi) \right] \bar{i}_z = - \nabla^2 \bar{H} = -\mu\sigma j\omega \hat{H}_\phi$$

$$\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \hat{H}_\phi) \right] - j\omega\mu\sigma \hat{H}_\phi = 0$$

$$r^2 \frac{d^2 \hat{H}_\phi}{dr^2} + r \frac{d\hat{H}_\phi}{dr} - \hat{H}_\phi (j\omega\mu\sigma r^2 + 1) = 0$$

$$\hat{H}_\phi(r) = A_1 J_1 \left[ \frac{r}{\delta} (1-j) \right] ; \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$\hat{H}_\phi(r=a) = \frac{I_o}{2\pi a} \rightarrow A_1 = \frac{I_o}{2\pi a J_1 \left[ \frac{a}{\delta} (1-j) \right]}$$

$$\hat{H}_\phi(r) = \frac{I_o}{2\pi a} \frac{J_1 \left[ \frac{r}{\delta} (1-j) \right]}{J_1 \left[ \frac{a}{\delta} (1-j) \right]}$$

$$c) \quad \hat{J}_z = \frac{1}{r} \frac{d}{dr} (r \hat{H}_\phi) = \frac{d\hat{H}_\phi}{dr} + \frac{\hat{H}_\phi}{r}$$

# ELECTROMAGNETIC INDUCTION

$$= A_1 \left[ \frac{dJ_1}{dr} + \frac{J_1}{r} \right]$$

$$= \frac{I_o}{2\pi a J_1 \left[ \frac{a}{\delta} (1-j) \right]} \left[ J_o \left[ \frac{r}{\delta} (1-j) \right] \right] \frac{(1-j)}{\delta}$$

## Section 6.6

33. a)  $T = \frac{1}{2} i^2 \frac{dL}{d\theta} = -L_1 i_1^2 \sin 2\theta = -L_1 I_o^2 \cos^2 \omega_o t \sin 2\theta$

b)  $T = -L_1 I_o^2 \left[ \frac{1}{2} \sin 2\theta + \frac{1}{4} \left[ \sin 2(\omega_o t + \theta) + \sin 2(\theta - \omega_o t) \right] \right]$

$$\theta = \omega t + \delta \Rightarrow \langle T \rangle = -\frac{L_1 I_o^2}{4} \sin 2\delta \text{ when } \omega = \pm \omega_o$$

c)  $\langle T \rangle_{\max} = -\frac{1}{4} L_1 I_o^2 \text{ when } \delta = \frac{\pi}{4}$

34. a)  $\Phi_1 = L_1(\theta) i_1 + M(\theta) i_2$

$$\Phi_2 = M(\theta) i_1 + L_2(\theta) i_2$$

$$v_1 = \frac{d\Phi_1}{dt}, \quad v_2 = \frac{d\Phi_2}{dt}$$

$$p = v_1 i_1 + v_2 i_2 = i_1 \frac{d\Phi_1}{dt} + i_2 \frac{d\Phi_2}{dt}$$

b)  $p = i_1 \frac{d}{dt} \left[ L_1(\theta) i_1 + M(\theta) i_2 \right] + i_2 \frac{d}{dt} \left[ M(\theta) i_1 + L_2(\theta) i_2 \right]$

$$= \frac{d}{dt} \left[ \frac{1}{2} L_1(\theta) i_1^2 + \frac{1}{2} L_2(\theta) i_2^2 + M(\theta) i_1 i_2 \right]$$



# ELECTROMAGNETIC INDUCTION

$$+ \frac{1}{2} \left[ i_1^2 \frac{dL_1}{d\theta} + i_2^2 \frac{dL_2}{d\theta} + 2i_1i_2 \frac{dM(\theta)}{d\theta} \right] \frac{d\theta}{dt}$$

$$W = \frac{1}{2} L_1(\theta) i_1^2 + \frac{1}{2} L_2(\theta) i_2^2 + M(\theta) i_1 i_2 \quad \text{energy stored}$$

$$T = \frac{1}{2} i_1^2 \frac{dL_1}{d\theta} + \frac{1}{2} i_2^2 \frac{dL_2}{d\theta} + i_1 i_2 \frac{dM}{d\theta} \quad \text{Torque}$$

c)  $L_1 = \text{constant}, L_2 = \text{constant}, M(\theta) = M_o \sin\theta$

$$T = i_1 i_2 \frac{dM}{d\theta} = M_o I_1 I_2 \cos\theta$$

d)  $v_1 = L_1 \frac{di_1}{dt} + M_o I_2 \cos\theta \frac{d\theta}{dt} = -i_1 R \quad ; \quad R = \frac{\ell}{\sigma A}$

e)  $J \frac{d^2\theta}{dt^2} = T = M_o i_1 I_2 \cos\theta$

Linearized:  $J \frac{d^2\theta}{dt^2} = M_o I_2 i_1$

$$M_o I_2 \frac{d\theta}{dt} = -i_1 R - L_1 \frac{di_1}{dt}$$

$$\theta = \hat{\theta} e^{st}, \quad i_1 = \hat{I}_1 e^{st}$$

$$\hat{\theta} \left[ J s^2 \right] - M_o I_2 \hat{I}_1 = 0$$

$$\hat{\theta} \left[ s M_o I_2 \right] + \hat{I}_1 \left[ R + L_1 s \right] = 0$$

$$\rightarrow J s^2 (R + L_1 s) + (M_o I_2)^2 s = 0$$

$$s = 0, \quad s^2 + \alpha s + \omega_o^2 = 0 \quad ; \quad \alpha = \frac{R}{L_1}, \quad \omega_o^2 = \frac{(M_o I_2)^2}{J L_1}$$

$$s = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \omega_o^2}$$

# ELECTROMAGNETIC INDUCTION

$$f) \quad \theta(t) = \left[ A_1 e^{j\beta t} + A_2 e^{-j\beta t} \right] e^{-\frac{\alpha}{2} t} + A_3 ; \quad \beta = \sqrt{\omega_o^2 - \left(\frac{\alpha}{2}\right)^2}$$

$$\theta(t \rightarrow \infty) = 0 \rightarrow A_3 = 0$$

$$\theta(t = 0) = \theta_o = A_1 + A_2$$

$$\left. \frac{d\theta}{dt} \right|_{t=0} = 0 = (j\beta - \frac{\alpha}{2}) A_1 - (j\beta + \frac{\alpha}{2}) A_2$$

$$A_1 = \frac{(j\beta + \frac{\alpha}{2}) \theta_o}{2j\beta}$$

$$A_2 = \frac{(j\beta - \frac{\alpha}{2}) \theta_o}{2j\beta}$$

$$\theta(t) = \frac{\theta_o}{2j\beta} \left[ (j\beta + \frac{\alpha}{2}) e^{j\beta t} + (j\beta - \frac{\alpha}{2}) e^{-j\beta t} \right] e^{-\frac{\alpha}{2} t}$$

$$= \theta_o \left[ \cos \beta t + \frac{\alpha}{2\beta} \sin \beta t \right] e^{-\frac{\alpha}{2} t}$$

$$i_1(t) = \frac{J}{M_o I_2} \frac{d^2 \theta}{dt^2} = \frac{-J \theta_o \omega_o^2}{M I_2 \beta} \left[ \beta \cos \beta t - \frac{\alpha}{2} \sin \beta t \right] e^{-\frac{\alpha}{2} t}$$

Solutions: oscillatory if  $\beta$  real  $\rightarrow \omega_o^2 > \frac{\alpha^2}{4}$

Damped if  $\beta$  imaginary  $\rightarrow \omega_o^2 < \frac{\alpha^2}{4}$

$$35. a) \quad L(x) = \frac{\mu_o x}{2\pi} \ln \frac{b}{a}$$

$$b) \quad f_x = \frac{1}{2} i^2 \frac{dL}{dx} = \frac{\mu_o i^2}{4\pi} \ln \frac{b}{a}$$

# ELECTROMAGNETIC INDUCTION

## 36. Magnetic Circuit

$$\text{In gaps } H = \frac{NI}{2x} \rightarrow \Phi = \frac{\mu_o NI}{2x} sD$$

$$\text{a) } L(x) = \frac{N\Phi}{I} = \frac{\mu_o N^2 sD}{2x} \rightarrow \text{(b) } f_x = \frac{1}{2} i^2 \frac{dL}{dx} = - \frac{I^2 \mu_o N^2 sD}{4x^2}$$

## Cylinder-plane

$$\text{a) } L = \frac{\mu_o D}{2\pi} \cosh^{-1} \frac{x+a}{a} \quad (\text{from Prob. 30})$$

$$\text{b) } f_x = \frac{1}{2} i^2 \frac{dL}{dx} = \frac{\mu_o D i^2}{4\pi} \frac{1}{a} \frac{1}{\left[\left(1+\frac{x^2}{a^2}\right) - 1\right]^{1/2}} = \frac{\mu_o i^2 D}{4\pi [x^2 + 2ax]^{1/2}}$$

$$37. \quad L(x) = \frac{1n \frac{b}{a}}{2\pi} [\mu_o (b-x) + \mu x]$$

$$f_x = \frac{1}{2} I^2 \frac{dL}{dx} = \frac{1n \frac{b}{a} I^2}{4\pi} [\mu - \mu_o] = \rho_m g h \pi (b^2 - a^2)$$

$$h = \frac{I^2 1n \frac{b}{a}}{4\pi^2 \rho_m g} (\mu - \mu_o)$$

## CHAPTER 7

### ELECTRODYNAMICS — FIELDS AND WAVES

#### Section 7.1

$$1. \quad \nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}; \quad \nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} + \bar{J}_f; \quad \nabla \cdot \bar{E} = \frac{\rho_f}{\epsilon}$$

$$a) \quad \bar{E} = E_o (\bar{x}\bar{i}_x + \bar{y}\bar{i}_y) \sin \omega t$$

$$\rho_f = \epsilon \nabla \cdot \bar{E} = 2\epsilon E_o \sin \omega t$$

$$\nabla \times \bar{E} = 0 \rightarrow \bar{H} = 0 \rightarrow \bar{J}_f = -\epsilon \frac{\partial \bar{E}}{\partial t} = -\epsilon E_o (\bar{x}\bar{i}_x + \bar{y}\bar{i}_y) \omega \cos \omega t$$

$$b) \quad \bar{E} = E_o (\bar{y}\bar{i}_x - \bar{x}\bar{i}_y) \cos \omega t$$

$$\rho_f = \epsilon \nabla \cdot \bar{E} = 0$$

$$\nabla \times \bar{E} = \bar{i}_z \left[ \frac{\partial E}{\partial x} - \frac{\partial E}{\partial y} \right] = -2E_o \cos \omega t \bar{i}_z = -\mu \frac{\partial \bar{H}}{\partial t} \rightarrow \bar{H} = \frac{2E_o}{\omega \mu} \sin \omega t \bar{i}_z$$

$$\bar{J}_f = \nabla \times \bar{H} - \epsilon \frac{\partial \bar{E}}{\partial t} = +\epsilon \omega E_o (\bar{y}\bar{i}_x - \bar{x}\bar{i}_y) \sin \omega t$$

$$c) \quad \bar{E} = \text{Re} E_o e^{j(\omega t - k_x x - k_z z)} \bar{i}_y$$

$$\rho_f = \epsilon \nabla \cdot \bar{E} = 0$$

$$\nabla \times \bar{E} = -\bar{i}_x \frac{\partial E}{\partial z} + \bar{i}_z \frac{\partial E}{\partial x} = \text{Re} [-jk_x \bar{i}_z + jk_z \bar{i}_x] E_o e^{j(\omega t - k_x x - k_z z)} = -\mu \frac{\partial \bar{H}}{\partial t}$$

$$\bar{H} = \text{Re} \frac{E_o}{\omega \mu} [k_x \bar{i}_z - k_z \bar{i}_x] e^{j(\omega t - k_x x - k_z z)}$$

$$\nabla \times \bar{H} = \bar{i}_y \left[ \frac{\partial H}{\partial z} - \frac{\partial H}{\partial x} \right] = \bar{i}_y \text{Re} \frac{jE_o}{\omega \mu} [k_z^2 + k_x^2] e^{j(\omega t - k_x x - k_z z)}$$

$$\bar{J}_f = \nabla \times \bar{H} - \epsilon \frac{\partial \bar{E}}{\partial t} = \bar{i}_y \text{Re} \left[ \frac{jE_o}{\omega \mu} (k_x^2 + k_z^2) - j\omega \epsilon E_o \right] e^{j(\omega t - k_x x - k_z z)}$$

$$\bar{J}_f = 0 \rightarrow k_x^2 + k_z^2 = \omega^2 \epsilon \mu$$

$$2. \quad \nabla \times \vec{H} = \vec{J}_f + \epsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon}$$

$$a) \quad \nabla \cdot \vec{J}_f + \frac{\partial \rho_f}{\partial t} = \frac{\partial \rho_f}{\partial t} + \frac{\rho_f}{\tau} = 0 \rightarrow \rho_f = \rho_o(\vec{r}, t=0) e^{-t/\tau}; \quad \tau = \epsilon/\sigma$$

$$b) \quad \rho_f = \rho_o e^{-t/\tau}; \quad \nabla \cdot \vec{E} = \frac{dE_x}{dx} = \frac{\rho_o}{\epsilon} e^{-t/\tau} \rightarrow E_x = \frac{\rho_o x}{\epsilon} e^{-t/\tau} + C(t)$$

$$\frac{i(t)}{A} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t} = C(t) = (\nabla \times \vec{H})_x$$

$$\text{Open Circuit: } C(t) = 0 = (\nabla \times \vec{H})_x \rightarrow \vec{H} = 0$$

Short Circuit:

$$\int_0^d E_x dx = 0 = \frac{\rho_o d^2}{2\epsilon} e^{-t/\tau} + C(t)d \rightarrow E_x = \frac{\rho_o}{2\epsilon} (x - \frac{d}{2}) e^{-t/\tau}$$

$$(\nabla \times \vec{H})_x = \frac{\partial H_z}{\partial y} = C(t) = -\frac{\rho_o d}{2\epsilon} e^{-t/\tau} \rightarrow H_z = -\frac{\rho_o dy}{2\epsilon} e^{-t/\tau}$$

$$c) \quad \rho_f = \rho_o e^{-t/\tau}$$

$$\nabla \cdot \vec{E} = \frac{1}{r} \frac{\partial}{\partial r} (r E_r) = \frac{\rho_o}{\epsilon} e^{-t/\tau} \rightarrow E_r = \frac{\rho_o r e^{-t/\tau}}{2\epsilon} + \frac{C(t)}{r}$$

$$\frac{i(t)}{2\pi r l} = \sigma E_r + \epsilon \frac{\partial E_r}{\partial t} = \frac{C(t)}{r} = (\nabla \times \vec{H})_r$$

$$\text{Open Circuit: } i(t) = 0 \rightarrow C(t) = 0 \rightarrow \vec{H} = 0$$

Short Circuit:

$$\int_a^b E_r dr = 0 = \frac{\rho_o (b^2 - a^2) e^{-t/\tau}}{4\epsilon} + C(t) \ln \frac{b}{a}$$

$$C(t) = -\frac{\rho_o (b^2 - a^2) e^{-t/\tau}}{4\epsilon \ln \frac{b}{a}}$$

$$(\nabla \times \vec{H})_r = -\left[\frac{1}{r} \frac{\partial H_\phi}{\partial z}\right] = \frac{C(t)}{r} \rightarrow H_\phi = -C(t)z$$

$$d) \quad \vec{H} = 0 \text{ when } i(t) = 0 \quad [\text{Open circuited terminals}]$$

$$3. \quad a) \quad \nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t}$$

$$\nabla \cdot \bar{B} = 0 \rightarrow \bar{B} = \nabla \times \bar{A} \rightarrow \nabla \times [\bar{E} + \frac{\partial \bar{A}}{\partial t}] = 0$$

$$b) \quad \bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla V$$

$$c) \quad \nabla \times \bar{H} = \frac{1}{\mu} \nabla \times (\nabla \times \bar{A}) = \frac{1}{\mu} [\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}] = \bar{J}_f + \epsilon \frac{\partial \bar{E}}{\partial t}$$

$$= \bar{J}_f - \epsilon [\frac{\partial^2 \bar{A}}{\partial t^2} + \nabla \frac{\partial V}{\partial t}]$$

$$\nabla^2 \bar{A} - \epsilon \mu \frac{\partial^2 \bar{A}}{\partial t^2} - \nabla [\nabla \cdot \bar{A} + \epsilon \mu \frac{\partial V}{\partial t}] = -\mu \bar{J}_f$$

$$d) \quad \nabla \cdot \bar{A} + \epsilon \mu \frac{\partial V}{\partial t} = 0$$

$$e) \quad \nabla \cdot \bar{E} = -\nabla^2 V - \frac{\partial}{\partial t} [\nabla \cdot \bar{A}] = \frac{\rho_f}{\epsilon}$$

$$\text{From (d)} \quad \nabla^2 V - \epsilon \mu \frac{\partial^2 V}{\partial t^2} = - \frac{\rho_f}{\epsilon}$$

$$f) \quad r > 0, \rho_f = 0$$

$$\nabla^2 V - \epsilon \mu \frac{\partial^2 V}{\partial t^2} = 0; \quad V(r, t) = \text{Re} \hat{V}(r) e^{j\omega t}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{d\hat{V}}{dr}) + \frac{\omega^2}{c^2} \hat{V} = 0; \quad c^2 = \frac{1}{\epsilon \mu}$$

$$\frac{\partial^2}{\partial r^2} (r\hat{V}) + k^2 (r\hat{V}) = 0; \quad k = \frac{\omega}{c}$$

$$r\hat{V} = V_1 e^{-jkr} + V_2 e^{+jkr} \quad \nearrow^0$$

$$\lim_{r \rightarrow 0} \hat{V} = \frac{\hat{Q}}{4\pi\epsilon r} = \frac{V_1}{r} \rightarrow \hat{V} = \frac{\hat{Q}}{4\pi\epsilon r} e^{-jkr}$$

### Section 7.2

$$4. \quad a) \quad \bar{H} \cdot [\nabla \times \bar{E}] = -\mu_0 \frac{\partial}{\partial t} [\bar{H} + \bar{M}]$$

$$-\vec{E} \cdot (\nabla \times \vec{H}) = \vec{J}_f + \frac{\partial}{\partial t} [\epsilon_0 \vec{E} + \vec{P}]$$

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) = -\frac{\partial}{\partial t} \left[ \frac{1}{2} \epsilon_0 |\vec{E}|^2 + \frac{1}{2} \mu_0 |\vec{H}|^2 \right] - \mu_0 \vec{H} \cdot \frac{\partial \vec{M}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} - \vec{E} \cdot \vec{J}_f$$

$$\vec{S} = \vec{E} \times \vec{H}, w = \frac{1}{2} \epsilon_0 |\vec{E}|^2 + \frac{1}{2} \mu_0 |\vec{H}|^2, P_d = \vec{E} \cdot \vec{J}_f, P_P = \vec{E} \cdot \frac{\partial \vec{P}}{\partial t}, P_M = \mu_0 \vec{H} \cdot \frac{\partial \vec{M}}{\partial t}$$

$$\nabla \cdot \vec{S} + \frac{\partial w}{\partial t} = -P_d - P_P - P_M$$

b) Energy dissipated per cycle equals area of hysteresis loops

$$w_P = \int P_P dt = \int \vec{E} \cdot d\vec{P} = 4P_S E_C$$

$$w_M = \int P_M dt = \int \mu_0 \vec{H} \cdot d\vec{M} = 4\mu_0 M_S H_C$$

$$w_T = w_P + w_M = 4[P_S E_C + \mu_0 M_S H_C]$$

$$5. a) \nabla \cdot (\vec{E} \times \vec{H}) + \frac{\partial}{\partial t} \left[ \frac{1}{2} \epsilon |\vec{E}|^2 + \frac{1}{2} \mu |\vec{H}|^2 \right] = -\vec{E} \cdot \vec{J}_f = \frac{-1}{\omega_P^2 \epsilon} \vec{J}_f \cdot \frac{\partial \vec{J}_f}{\partial t}$$

$$\nabla \cdot (\vec{E} \times \vec{H}) + \frac{\partial}{\partial t} \left[ \frac{1}{2} \epsilon |\vec{E}|^2 + \frac{1}{2} \mu |\vec{H}|^2 + \frac{1}{2\omega_P^2 \epsilon} |\vec{J}_f|^2 \right] = 0$$

$$\vec{S} = \vec{E} \times \vec{H}, w = \frac{1}{2} \epsilon |\vec{E}|^2 + \frac{1}{2} \mu |\vec{H}|^2 + \frac{1}{2\omega_P^2 \epsilon} |\vec{J}_f|^2$$

$$b) \vec{J}_f = qn\vec{v} \rightarrow \frac{1}{2\omega_P^2 \epsilon} |\vec{J}_f|^2 = \frac{1}{2\omega_P^2 \epsilon} q^2 n^2 |\vec{v}|^2 = \frac{1}{2} nm |\vec{v}|^2$$

c) Kinetic energy

$$d) \nabla \times \hat{\vec{E}} = -j\omega\mu\hat{\vec{H}}$$

$$\nabla \times \hat{\vec{H}} = j\omega\epsilon \left(1 - \frac{\omega_P^2}{\omega^2}\right) \hat{\vec{E}}$$

$$\nabla \cdot \left[ \frac{1}{2} \hat{\vec{E}} \times \hat{\vec{H}}^* \right] = \frac{1}{2} [\hat{\vec{H}}^* \cdot (\nabla \times \hat{\vec{E}}) - \hat{\vec{E}} \cdot (\nabla \times \hat{\vec{H}}^*)]$$

$$= \frac{1}{2} \hat{\mathbf{H}}^* \cdot [-j\omega\mu\hat{\mathbf{H}}] + \frac{1}{2} \hat{\mathbf{E}} \cdot [j\omega\epsilon(1 - \frac{\omega_P^2}{\omega^2})\hat{\mathbf{E}}^*]$$

$$= j\omega[\frac{1}{2} \epsilon |\hat{\mathbf{E}}|^2 (1 - \frac{\omega_P^2}{\omega^2}) - \frac{1}{2} \mu |\hat{\mathbf{H}}|^2]$$

$$e) \nabla \cdot [\frac{1}{2} \hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] + 2j\omega\langle w \rangle = 0 \rightarrow \langle w \rangle = -\frac{1}{4} \epsilon |\hat{\mathbf{E}}|^2 (1 - \frac{\omega_P^2}{\omega^2}) + \frac{1}{4} \mu |\hat{\mathbf{H}}|^2$$

$$6. a) \mathbf{E}_x = \frac{V}{s}, \mathbf{H}_y = \frac{NI}{\ell}$$

$$S_z = \mathbf{E}_x \mathbf{H}_y = \frac{NVI}{\ell s}$$

$$b) w = \frac{1}{2} \mu H_y^2 + \frac{1}{2} \epsilon E_x^2 = \frac{1}{2} \mu [\frac{NI}{\ell}]^2 + \frac{1}{2} \epsilon [\frac{V}{s}]^2$$

$$c) \nabla \cdot \bar{\mathbf{S}} = 0, \frac{\partial w}{\partial t} = 0, \nabla \cdot \bar{\mathbf{S}} + \frac{\partial w}{\partial t} = 0$$

$$7. \hat{\mathbf{E}}(\mathbf{r}) = \bar{\mathbf{E}}_r + j\bar{\mathbf{E}}_i$$

$$a) \hat{\mathbf{E}} \cdot \hat{\mathbf{E}} = |\bar{\mathbf{E}}_r|^2 - |\bar{\mathbf{E}}_i|^2 + 2j \bar{\mathbf{E}}_r \cdot \bar{\mathbf{E}}_i$$

$$\hat{\mathbf{E}} \cdot \hat{\mathbf{E}} = 0 \text{ if } |\bar{\mathbf{E}}_r| = |\bar{\mathbf{E}}_i| \text{ and } \bar{\mathbf{E}}_r \perp \bar{\mathbf{E}}_i$$

$$b) \hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^* = |\bar{\mathbf{E}}_r|^2 + |\bar{\mathbf{E}}_i|^2 \quad [\text{Never zero}]$$

$$c) \hat{\mathbf{E}} \times \hat{\mathbf{E}} = [\bar{\mathbf{E}}_r + j\bar{\mathbf{E}}_i] \times [\bar{\mathbf{E}}_r + j\bar{\mathbf{E}}_i] = j[\bar{\mathbf{E}}_r \times \bar{\mathbf{E}}_i + \bar{\mathbf{E}}_i \times \bar{\mathbf{E}}_r] = 0 \quad [\text{Always zero}]$$

$$d) \hat{\mathbf{E}} \times \hat{\mathbf{E}}^* = [\bar{\mathbf{E}}_r + j\bar{\mathbf{E}}_i] \times [\bar{\mathbf{E}}_r - j\bar{\mathbf{E}}_i] = -j[\bar{\mathbf{E}}_r \times \bar{\mathbf{E}}_i - \bar{\mathbf{E}}_i \times \bar{\mathbf{E}}_r] = 2j\bar{\mathbf{E}}_i \times \bar{\mathbf{E}}_r$$

$$[\text{Zero if } \bar{\mathbf{E}}_r \parallel \bar{\mathbf{E}}_i]$$

### Section 7.3

$$8. a) \mathbf{E}_x, \mathbf{H}_y$$

$$\nabla \times \bar{\mathbf{E}} = -\mu \frac{\partial \bar{\mathbf{H}}}{\partial t} \rightarrow \frac{\partial \bar{\mathbf{E}}_x}{\partial z} = -\mu \frac{\partial \bar{\mathbf{H}}_y}{\partial t}$$

$$\nabla \times \bar{\mathbf{H}} = \sigma \bar{\mathbf{E}} + \epsilon \frac{\partial \bar{\mathbf{E}}}{\partial t} \rightarrow \frac{\partial \bar{\mathbf{H}}_y}{\partial z} = -[\sigma \bar{\mathbf{E}}_x + \epsilon \frac{\partial \bar{\mathbf{E}}_x}{\partial t}]$$



$$b) \quad \epsilon\mu \frac{\partial^2 E_x}{\partial t^2} + \mu\sigma \frac{\partial E_x}{\partial t} - \frac{\partial^2 E_x}{\partial z^2} = 0$$

$$c) \quad \hat{E}_x(z) = \hat{E} e^{-jkz} \rightarrow k^2 = \frac{\omega^2}{c^2} - j\omega\mu\sigma; \quad c^2 = \frac{1}{\epsilon\mu}$$

$$\hat{E}_x(z) = \hat{E}_1 e^{-jkz} + \hat{E}_2 e^{+jkz}; \quad k = \sqrt{\frac{\omega^2}{c^2} + j\omega\mu\sigma}$$

$$d) \quad i) \quad \frac{\sigma}{\omega\epsilon} \ll 1, k = \frac{\omega}{c} \sqrt{1 - \frac{j\sigma}{\omega\epsilon}} \approx \frac{\omega}{c} \left[1 - \frac{1}{2} \frac{j\sigma}{\omega\epsilon}\right] \approx \frac{\omega}{c} - \frac{j}{2} \sigma \sqrt{\frac{\mu}{\epsilon}}$$

$$ii) \quad \frac{\sigma}{\omega\epsilon} \gg 1, k \approx \sqrt{-j\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}} (1 - j) = \frac{(1 - j)}{\delta}$$

$$e) \quad \nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}_f + \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow \frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon \left[ \omega_P^2 E_x + \frac{\partial^2 E_x}{\partial t^2} \right]$$

$$\frac{\partial^2 E_x}{\partial t^2} + \omega_P^2 E_x - c^2 \frac{\partial^2 E_x}{\partial z^2} = 0; \quad E_x(z, t) = \text{Re} \hat{E}_x e^{j(\omega t - kz)}$$

$$k^2 = \frac{\omega^2 - \omega_P^2}{c^2} \rightarrow k = \pm \sqrt{\frac{\omega^2 - \omega_P^2}{c^2}}$$

$$f) \quad H_y = \begin{cases} -\frac{K_0}{2} \text{Re} e^{j(\omega t - kz)} \\ \frac{K_0}{2} \text{Re} e^{j(\omega t + kz)} \end{cases}, \quad E_x = \frac{\omega\mu H_y}{\pm k} = \begin{cases} -\frac{K_0}{2} \text{Re} \frac{\omega\mu}{k} e^{j(\omega t - kz)} \\ -\frac{K_0}{2} \text{Re} \frac{\omega\mu}{k} e^{j(\omega t + kz)} \end{cases}$$

$$9. \quad a) \quad \nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}_f + \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow -\frac{\partial H_y}{\partial z} = J_x + \epsilon \frac{\partial E_x}{\partial t}$$

$$\epsilon \frac{\partial^2 E_x}{\partial t^2} + \frac{\partial J_x}{\partial t} = -\frac{\partial^2 H_y}{\partial z \partial t} = \frac{1}{\mu} \frac{\partial^2 E_x}{\partial z^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} = \mu \frac{\partial J_x}{\partial t}; \quad c^2 = \frac{1}{\epsilon \mu}$$

b) Homogeneous Solution:  $E_x = [\text{Re } \hat{E}_1 e^{-jkz} + \hat{E}_2 e^{jkz}] e^{j\omega t}; \quad k = \frac{\omega}{c}$

Particular Solution:  $E_x = \text{Re} \left[ \frac{\mu c^2 j}{\omega} J_o e^{j\omega t} \right]$

$$\hat{E}_x(z) = \begin{cases} \hat{E}_1 e^{-jk_o(z-d)} & z > d; \quad k_o = \omega \sqrt{\epsilon_o \mu_o} \\ \hat{E}_2 e^{-jkz} + \hat{E}_3 e^{jkz} + \frac{jJ_o}{\omega \epsilon} & -d < z < d; \quad k = \omega \sqrt{\epsilon \mu} \\ \hat{E}_4 e^{+jk_o(z+d)} & z < -d \end{cases}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu} \frac{\partial E_x}{\partial z} \rightarrow \hat{H}_y(z) = \begin{cases} \frac{\hat{E}_1}{\eta_o} e^{-jk_o(z-d)} & z > d; \quad \eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \\ \frac{1}{\eta} [\hat{E}_2 e^{-jkz} - \hat{E}_3 e^{jkz}] & -d < z < d; \quad \eta = \sqrt{\frac{\mu}{\epsilon}} \\ -\frac{\hat{E}_4}{\eta_o} e^{jk_o(z+d)} & z < -d \end{cases}$$

Boundary Conditions:

$$E_x(z=d_+) = E_x(z=d_-) \rightarrow \hat{E}_1 = \hat{E}_2 e^{-jkd} + \hat{E}_3 e^{jkd} + \frac{jJ_o}{\omega \epsilon}$$

$$H_y(z=d_+) = H_y(z=d_-) \rightarrow \frac{\hat{E}_1}{\eta_o} = \frac{1}{\eta} [\hat{E}_2 e^{-jkd} - \hat{E}_3 e^{jkd}]$$

$$E_x(z=-d_+) = E_x(z=-d_-) \rightarrow \hat{E}_4 = \hat{E}_2 e^{jkd} + \hat{E}_3 e^{-jkd} + \frac{jJ_o}{\omega \epsilon}$$

$$H_y(z=-d_+) = H_y(z=-d_-) \rightarrow -\frac{\hat{E}_4}{\eta_o} = \frac{1}{\eta} [\hat{E}_2 e^{jkd} - \hat{E}_3 e^{-jkd}]$$

Symmetry  $\rightarrow \hat{E}_1 = \hat{E}_4, \hat{E}_2 = \hat{E}_3$

$$\hat{E}_1 = \hat{E}_4 = \frac{j\eta_o J_o \sin kd}{\omega \epsilon [\eta_o \sin kd - j\eta \cos kd]}; \quad \hat{E}_2 = \hat{E}_3 = -\frac{J_o \eta}{2\omega \epsilon [\eta_o \sin kd - j\eta \cos kd]}$$

$$\hat{E}_x(z) = \begin{cases} \frac{j\eta_o J_o \text{sinkd} e^{-jk_o(z-d)}}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} & z > d \\ \frac{-J_o \eta \cos kz}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} + \frac{jJ_o}{\omega\epsilon} & -d < z < d \\ \frac{j\eta_o J_o \text{sinkd} e^{+jk_o(z+d)}}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} & z < -d \end{cases}$$

$$\hat{H}_y(z) = \begin{cases} \frac{jJ_o \text{sinkd} e^{-jk_o(z-d)}}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} & z > d \\ \frac{jJ_o \text{sink} z}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} & -d < z < d \\ \frac{-jJ_o \text{sinkd} e^{jk_o(z+d)}}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} & z < -d \end{cases}$$

$$c) P_V = -\frac{1}{2} \text{Re} \int_{-d}^d \hat{E} \cdot \hat{J}_f^* dz$$

$$= \text{Re} \left[ \frac{1}{2} \frac{J_o^2 \eta}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} \int_{-d}^d \cos kz dz + \frac{jJ_o^2}{\omega\epsilon} \right]$$

$$= \text{Re} \left[ \frac{J_o^2 \eta \text{sinkd}}{k\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} \right]$$

$$= \frac{J_o^2 \eta_o \eta \sin^2 kd}{k\omega\epsilon[(\eta_o \text{sinkd})^2 + (\eta \cos kd)^2]}$$

$$d) P_z(z=d) = \frac{1}{2} \text{Re}(\hat{E}_x(d) \hat{H}_y^*(d))$$

$$= \frac{1}{2} \text{Re} \left[ \frac{j\eta_o J_o \text{sinkd}}{\omega\epsilon[\eta_o \text{sinkd} - j\eta \coskd]} \frac{(-jJ_o \text{sinkd})}{\omega\epsilon[\eta_o \text{sinkd} + j\eta \coskd]} \right]$$

$$= \frac{1}{2} \frac{\eta_o J_o^2 \sin^2 kd}{(\omega\epsilon)^2 [(\eta_o \text{sinkd})^2 + (\eta \cos kd)^2]}$$

$$k = \omega \sqrt{\epsilon \mu} \rightarrow \frac{\eta}{k \omega \epsilon} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\omega^2 \epsilon \sqrt{\epsilon \mu}} = \frac{1}{(\omega \epsilon)^2}$$

$$P_z(z=d) = -P_z(z=-d) = \frac{1}{2} P_V$$

$$10. \quad a) \quad \nabla \times \bar{E} = -\mu(z) \frac{\partial \bar{H}}{\partial t} \rightarrow \frac{\partial E_x}{\partial z} = -\mu(z) \frac{\partial H_y}{\partial t}$$

$$\nabla \times \bar{H} = \epsilon(z) \frac{\partial \bar{E}}{\partial t} \rightarrow \frac{\partial H_y}{\partial z} = -\epsilon(z) \frac{\partial E_x}{\partial t}$$

$$\frac{\partial}{\partial z} \left( \frac{1}{\epsilon(z)} \frac{\partial H_y}{\partial z} \right) = - \frac{\partial^2 E_x}{\partial z \partial t} = \mu(z) \frac{\partial^2 H_y}{\partial t^2}$$

$$\frac{\partial^2 H_y}{\partial t^2} - \frac{1}{\mu(z)} \frac{\partial}{\partial z} \left[ \frac{1}{\epsilon(z)} \frac{\partial H_y}{\partial z} \right] = 0$$

$$\frac{\partial}{\partial z} \left( \frac{1}{\mu(z)} \frac{\partial E_x}{\partial z} \right) = - \frac{\partial^2 H_y}{\partial z \partial t} = \epsilon(z) \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\epsilon(z)} \frac{\partial}{\partial z} \left[ \frac{1}{\mu(z)} \frac{\partial E_x}{\partial z} \right] = 0$$

$$b) \quad \epsilon(z) = \epsilon e^{\alpha|z|}, \quad \mu(z) = \mu e^{-\alpha|z|}$$

$$\frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\epsilon(z)\mu(z)} \frac{\partial^2 E_x}{\partial z^2} + \frac{1}{\epsilon(z)\mu^2(z)} \frac{\partial E_x}{\partial z} \frac{d\mu}{dz} = 0$$

$$\frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\epsilon\mu} \frac{\partial^2 E_x}{\partial z^2} + \frac{\alpha}{\epsilon\mu} \frac{\partial E_x}{\partial z} = 0 \quad \begin{matrix} z > 0 \\ z < 0 \end{matrix}$$

$$E_x(z,t) = \text{Re} \hat{E}_x(z) e^{j(\omega t - k|z|)}$$

$$-\omega^2 + \frac{k^2}{\epsilon\mu} + \frac{\alpha j k}{\epsilon\mu} = 0 \rightarrow k^2 + \alpha j k - \omega^2 \epsilon\mu = 0$$

$$k = -\frac{j\alpha}{2} \pm \sqrt{-\frac{\alpha^2}{4} + \omega^2 \epsilon\mu} = -\frac{j\alpha}{2} \pm \beta; \quad \beta = \sqrt{\omega^2 \epsilon\mu - \frac{\alpha^2}{4}}$$

$$E_x = \begin{cases} E_o e^{j(\omega t - \beta z)} e^{-\frac{\alpha}{2} z} & z > 0 \\ E_o e^{j(\omega t + \beta z)} e^{+\frac{\alpha}{2} z} & z < 0 \end{cases}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu(z)} \frac{\partial E_x}{\partial z} \rightarrow H_y = \begin{cases} \frac{(j\beta + \frac{\alpha}{2})}{j\omega\mu} E_o e^{j(\omega t - \beta z)} e^{+\frac{\alpha}{2} z} & z > 0 \\ \frac{-(j\beta + \frac{\alpha}{2})}{j\omega\mu} E_o e^{j(\omega t + \beta z)} e^{-\frac{\alpha}{2} z} & z < 0 \end{cases}$$

$$H_y(z=0_-) - H_y(z=0_+) = K_o e^{j\omega t} \rightarrow \frac{-2(j\beta + \frac{\alpha}{2})}{j\omega\mu} E_o = K_o \rightarrow E_o = \frac{j\omega\mu K_o}{-2(j\beta + \frac{\alpha}{2})}$$

c)  $\omega^2 \epsilon \mu - \frac{\alpha^2}{4} > 0$  oscillatory

$\omega^2 \epsilon \mu - \frac{\alpha^2}{4} < 0$  evanescent

11. a)  $z' = \gamma(z - vt)$ ,  $t' = \gamma(t - vz/c_o^2)$ ;  $\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c_o})^2}}$

$$z'_1 = \gamma(z_1 - vt), \quad z'_2 = \gamma(z_2 - vt)$$

$$t'_1 = \gamma(t_1 - vz_1/c_o^2), \quad t'_2 = \gamma(t_2 - vz_2/c_o^2)$$

$$t'_1 - t'_2 = \frac{\gamma v}{c_o^2} (z_2 - z_1)$$

b)  $t'_1 = \gamma(t_1 - \frac{vz_1}{c_o^2})$ ,  $t'_2 = \gamma(t_2 - \frac{vz_2}{c_o^2})$

$$t'_1 - t'_2 = \gamma(t_1 - t_2)$$

c)  $z'_2 - z'_1 = \gamma(z_2 - z_1) = \gamma L$

$$12. \text{ a) } \bar{u} = u_x \bar{i}_x + u_y \bar{i}_y + u_z \bar{i}_z; z' = \gamma(z - vt), t' = \gamma(t - \frac{vz}{c_o^2}); \gamma = \frac{1}{\sqrt{1 - (\frac{v}{c_o})^2}}$$

$$\bar{u}' = \frac{d\bar{r}'}{dt'}$$

$$u'_z = \frac{\Delta z'}{\Delta t'} = \frac{\Delta z - v\Delta t}{\Delta t - \frac{v\Delta z}{c_o^2}} = \frac{\frac{\Delta z}{\Delta t} - v}{1 - \frac{v}{c_o^2} \frac{\Delta z}{\Delta t}} = \frac{u_z - v}{1 - \frac{vu_z}{c_o^2}}$$

$$u'_x = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\gamma(\Delta t - \frac{v\Delta z}{c_o^2})} = \frac{\Delta x/\Delta t}{\gamma(1 - \frac{v}{c_o^2} \frac{\Delta z}{\Delta t})} = \frac{u_x \sqrt{1 - (\frac{v}{c_o})^2}}{1 - \frac{vu_z}{c_o^2}}$$

$$u'_y = \frac{u_y \sqrt{1 - (\frac{v}{c_o})^2}}{1 - \frac{vu_z}{c_o^2}}$$

$$\text{b) i) } \bar{u} = c_o \bar{i}_x \rightarrow u'_x = c_o \sqrt{1 - (\frac{v}{c_o})^2}, u'_y = 0, u'_z = -v$$

$$\text{ii) } \bar{u} = c_o \bar{i}_y \rightarrow u'_x = 0, u'_y = c_o \sqrt{1 - (\frac{v}{c_o})^2}, u'_z = -v$$

$$\text{iii) } \bar{u} = c_o \bar{i}_z \rightarrow u'_x = 0, u'_y = 0, u'_z = \frac{c_o - v}{1 - \frac{v}{c_o}} = c_o$$

$$\text{iv) } \bar{u} = \frac{c_o}{\sqrt{3}} [\bar{i}_x + \bar{i}_y + \bar{i}_z] \rightarrow u'_x = u'_y = \frac{c_o}{\sqrt{3}} \sqrt{1 - (\frac{v}{c_o})^2}, u'_z = \frac{\frac{c_o}{\sqrt{3}} - v}{1 - \frac{v}{c_o \sqrt{3}}}$$

$$\begin{aligned} \text{c) } \sqrt{u'^2_x + u'^2_y + u'^2_z} &= \frac{[(u_z - v)^2 + (u_x^2 + u_y^2)(1 - (\frac{v}{c_o})^2)]^{1/2}}{1 - \frac{vu_z}{c_o^2}} \\ &= \frac{\{u_x^2 + u_y^2 + u_z^2 - (\frac{v}{c_o})^2[u_x^2 + u_y^2 - c_o^2] - 2vu_z\}^{1/2}}{1 - \frac{vu_z}{c_o^2}} \end{aligned}$$

$$\text{If } u_x^2 + u_y^2 + u_z^2 = c_o^2$$

$$\sqrt{u_x'^2 + u_y'^2 + u_z'^2} = \frac{\{c_o^2 + (\frac{vu_z}{c_o})^2 - 2vu_z\}^{1/2}}{1 - \frac{vu_z}{c_o}}$$

$$= \frac{c_o - \frac{vu_z}{c_o}}{1 - \frac{vu_z}{c_o}} = c_o$$

#### Section 7.4

$$13. \quad \vec{E} = 100e^{j(2\pi \times 10^6 t - 2\pi \times 10^{-2} z)} \vec{i}_x$$

$$a) \quad \omega = 2\pi \times 10^6 \text{ rad/sec} \rightarrow f = \frac{\omega}{2\pi} = 10^6 \text{ Hz}$$

$$k = 2\pi \times 10^{-2} \text{ m}^{-1} \rightarrow \lambda = \frac{2\pi}{k} = 10^2 \text{ meter}$$

$$c = \frac{\omega}{k} = f\lambda = 10^8 \text{ meter/sec}$$

$$b) \quad c^2 = \frac{1}{\epsilon \mu_o} \rightarrow \epsilon = \frac{1}{c^2 \mu_o} = \frac{10^{-9}}{4\pi} = \epsilon_r \epsilon_o = \frac{\epsilon_r \times 10^{-9}}{36\pi} \rightarrow \epsilon_r = 9$$

$$\eta = \sqrt{\frac{\mu_o}{\epsilon}} = 40\pi$$

$$H_y = \frac{E_x}{\eta} = \frac{2.5}{\pi} e^{j(2\pi \times 10^6 t - 2\pi \times 10^{-2} z)} \text{ amps/meter}$$

$$c) \quad \langle S \rangle_z = \frac{1}{2} \frac{|\vec{E}|^2}{\eta} = \frac{10^4}{80\pi} = \frac{125}{\pi} \text{ watts/meter}^2$$

$$14. \quad \vec{E} = \text{Re}[E_{xo} \vec{i}_x + E_{yo} e^{j\phi} \vec{i}_y] e^{j(\omega t - kz)} = E_{xo} \cos(\omega t - kz) \vec{i}_x + E_{yo} \cos(\omega t - kz + \phi) \vec{i}_y$$

$$a) \quad \vec{H} = \frac{1}{\eta} \text{Re}[E_{xo} \vec{i}_y - E_{yo} e^{j\phi} \vec{i}_x] e^{j(\omega t - kz)}$$

$$= \frac{E_{xo}}{\eta} \cos(\omega t - kz) \vec{i}_y - \frac{E_{yo}}{\eta} \cos(\omega t - kz + \phi) \vec{i}_x$$

$$b) \quad \vec{S} = \vec{E} \times \vec{H} = \left[ \frac{E_{x0}^2}{\eta} \cos^2(\omega t - kz) + \frac{E_{y0}^2}{\eta} \cos^2(\omega t - kz + \phi) \right] \vec{i}_z$$

$$\langle \vec{S} \rangle = \frac{1}{2} \left[ \frac{E_{x0}^2 + E_{y0}^2}{\eta} \right] \vec{i}_z$$

$$15. \quad a) \quad M \frac{d^2 \bar{d}}{dt^2} + \frac{Q^2}{4\pi\epsilon_0 R_0^3} \bar{d} = QE_0 \cos \omega t \rightarrow \frac{d^2 \bar{d}}{dt^2} + \omega_0^2 \bar{d} = \frac{QE_0}{M} \cos \omega t; \quad \omega_0^2 = \frac{Q^2}{M4\pi\epsilon_0 R_0^3}$$

$$\bar{d} = \frac{QE_0}{M(\omega_0^2 - \omega^2)} \cos \omega t$$

$$b) \quad \vec{P} = NQ\bar{d} = \frac{NQ^2 E_0}{M(\omega_0^2 - \omega^2)} \cos \omega t = \frac{\epsilon_0 E_0 \omega_p^2}{\omega_0^2 - \omega^2} \cos \omega t; \quad \omega_p^2 = \frac{Q^2 N}{M\epsilon_0}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \rightarrow \vec{D} = \epsilon_0 \left[ 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right] E_0 \cos \omega t \rightarrow \epsilon(\omega) = \epsilon_0 \left[ 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right]$$

$$c) \quad k^2 = \omega^2 \epsilon(\omega) \mu_0 = \frac{\omega^2}{c^2} \left[ 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right]; \quad c^2 = \frac{1}{\epsilon_0 \mu_0}$$

$$d) \quad \text{Propagation} \rightarrow k \text{ real} \rightarrow \omega < \sqrt{\omega_0^2 + \omega_p^2}$$

$$\text{Evanesence} \rightarrow k \text{ imaginary} \rightarrow \omega > \sqrt{\omega_0^2 + \omega_p^2}$$

$$e) \quad v_p = \frac{\omega}{k} = \frac{c}{\left[ 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right]^{1/2}}$$

$$\begin{aligned} 2k \frac{dk}{d\omega} &= \frac{1}{c^2} \left[ 2\omega \left[ 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right] + \frac{\omega^2 \omega_p^2 2\omega}{(\omega_0^2 - \omega^2)^2} \right] \\ &= \frac{2\omega}{c^2} \left[ \frac{(\omega_0^2 - \omega^2)^2 + \omega_p^2 (\omega_0^2 - \omega^2) + \omega^2 \omega_p^2}{(\omega_0^2 - \omega^2)^2} \right] \\ &= \frac{2\omega}{c^2 (\omega_0^2 - \omega^2)^2} [(\omega_0^2 - \omega^2)^2 + \omega_p^2 \omega_0^2] \end{aligned}$$



$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} \frac{(\omega_o^2 - \omega^2)^2}{[(\omega_o^2 - \omega^2)^2 + \omega_p^2 \omega_o^2]}$$

$$f) \quad \nabla \times \hat{\mathbf{E}} = -j\omega\mu\hat{\mathbf{H}}$$

$$\nabla \times \hat{\mathbf{H}}^* = -j\omega\epsilon(\omega)\hat{\mathbf{E}}^*$$

$$\nabla \cdot [\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] = \hat{\mathbf{H}}^* \cdot (\nabla \times \hat{\mathbf{E}}) - \hat{\mathbf{E}} \cdot (\nabla \times \hat{\mathbf{H}}^*)$$

$$= -j\omega\mu|\hat{\mathbf{H}}|^2 + j\omega\epsilon(\omega)|\hat{\mathbf{E}}|^2$$

$$= -j\omega[\mu|\hat{\mathbf{H}}|^2 - \epsilon(\omega)|\hat{\mathbf{E}}|^2]$$

$$\nabla \cdot [\frac{1}{2} \hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] + 2j\omega[\frac{1}{4} \mu|\hat{\mathbf{H}}|^2 - \frac{1}{4} \epsilon(\omega)|\hat{\mathbf{E}}|^2] = 0$$

$$16. \quad a) \text{ and } b) \quad m \frac{dv_x}{dt} = q(E_x + v_y B_z) \rightarrow j\omega \hat{v}_x = \frac{q}{m} \hat{E}_x + \omega_o \hat{v}_y; \quad \omega_o = \frac{qB_z}{m}$$

$$m \frac{dv_y}{dt} = q(E_y - v_x B_z) \rightarrow j\omega \hat{v}_y = \frac{q}{m} \hat{E}_y - \omega_o \hat{v}_x$$

$$m \frac{dv_z}{dt} = qE_z \rightarrow j\omega \hat{v}_z = \frac{q}{m} \hat{E}_z$$

$$\hat{v}_x = \frac{(\frac{q}{m})[j\omega \hat{E}_x + \omega_o \hat{E}_y]}{\omega_o^2 - \omega^2}; \quad \hat{v}_y = \frac{(\frac{q}{m})[j\omega \hat{E}_y - \omega_o \hat{E}_x]}{\omega_o^2 - \omega^2}; \quad \hat{v}_z = \frac{q}{j\omega m} \hat{E}_z$$

$$\hat{J}_x = qn\hat{v}_x = \frac{\epsilon \omega_p^2}{(\omega_o^2 - \omega^2)} [j\omega \hat{E}_x + \omega_o \hat{E}_y]; \quad \omega_p^2 = \frac{q^2 n}{m\epsilon}$$

$$\hat{J}_y = qn\hat{v}_y = \frac{\epsilon \omega_p^2}{(\omega_o^2 - \omega^2)} [j\omega \hat{E}_y - \omega_o \hat{E}_x]$$

$$\hat{J}_z = qn\hat{v}_z = \frac{\epsilon \omega_p^2}{j\omega} \hat{E}_z$$

$$c) \quad \nabla \times \bar{\mathbf{E}} = -\mu \frac{\partial \bar{\mathbf{H}}}{\partial t} \rightarrow \begin{cases} \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \rightarrow jk\hat{E}_x = j\omega\mu\hat{H}_y \\ \frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \rightarrow -jk\hat{E}_y = j\omega\mu\hat{H}_x \end{cases}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \rightarrow \begin{cases} \frac{\partial H_y}{\partial z} = -J_x - \epsilon \frac{\partial E_x}{\partial t} \rightarrow jk\hat{H}_y = j\omega\epsilon\hat{E}_x + \frac{\epsilon\omega_p^2}{(\omega_o^2 - \omega^2)} [j\omega\hat{E}_x + \omega_o\hat{E}_y] \\ \frac{\partial H_x}{\partial z} = J_y + \epsilon \frac{\partial E_y}{\partial t} \rightarrow -jk\hat{H}_x = j\omega\epsilon\hat{E}_y + \frac{\epsilon\omega_p^2}{(\omega_o^2 - \omega^2)} [j\omega\hat{E}_y - \omega_o\hat{E}_x] \end{cases}$$

$$jk\hat{H}_y = \frac{jk^2}{\omega\mu} \hat{E}_x = j\omega\epsilon\hat{E}_x \left[ 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \right] + \frac{\epsilon\omega_p^2\omega_o}{(\omega_o^2 - \omega^2)} \hat{E}_y$$

$$-jk\hat{H}_x = \frac{jk^2}{\omega\mu} \hat{E}_y = j\omega\epsilon\hat{E}_y \left[ 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \right] - \frac{\epsilon\omega_p^2\omega_o}{(\omega_o^2 - \omega^2)} \hat{E}_x$$

$$\hat{E}_x \left[ \frac{k^2 c^2}{\omega^2} - \left( 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \right) \right] + \frac{j\omega_p^2\omega_o}{(\omega_o^2 - \omega^2)\omega} \hat{E}_y = 0; \quad c^2 = \frac{1}{\epsilon\mu}$$

$$-\hat{E}_x \frac{j\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)} + \hat{E}_y \left[ \frac{k^2 c^2}{\omega^2} - \left( 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \right) \right] = 0$$

$$\left[ \frac{k^2 c^2}{\omega^2} - \left( 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \right) \right]^2 - \left[ \frac{\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)} \right]^2 = 0$$

$$k^2 = \frac{\omega^2}{c^2} \left\{ 1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \pm \frac{\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)} \right\}$$

$$= \frac{\omega^2}{c^2} \left\{ 1 + \frac{\omega_p^2}{(\omega_o^2 - \omega^2)} \left( 1 \pm \frac{\omega_o}{\omega} \right) \right\}$$

$$= \frac{\omega^2}{c^2} \left\{ 1 + \frac{\omega_p^2}{(\omega_o^2 - \omega^2)} \left( \frac{\omega \pm \omega_o}{\omega} \right) \right\}$$

$$= \frac{\omega^2}{c^2} \left\{ 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_o)} \right\}$$

$$d) \quad k = 0 \rightarrow \omega^2 \mp \omega\omega_o - \omega_p^2 = 0 \rightarrow \omega = \pm \frac{\omega_o}{2} \pm \sqrt{\left(\frac{\omega_o}{2}\right)^2 + \omega_p^2}$$

$$k \rightarrow \infty \rightarrow \omega = \pm \omega_0$$

$$\text{Mode (1)} \quad k^2 = \frac{\omega^2}{c^2} \left\{ \frac{\omega^2 - \omega\omega_0 - \omega_p^2}{\omega(\omega - \omega_0)} \right\}$$

$$\text{Mode (2)} \quad k^2 = \frac{\omega^2}{c^2} \left\{ \frac{\omega^2 + \omega\omega_0 - \omega_p^2}{\omega(\omega + \omega_0)} \right\}$$

$$\text{Propagation} \rightarrow k \text{ real} \rightarrow \begin{cases} \text{Mode (1)} & \omega > \frac{\omega_0}{2} + \sqrt{\left(\frac{\omega_0}{2}\right)^2 + \omega_p^2}, & \omega < \omega_0 \\ \text{Mode (2)} & \omega > -\frac{\omega_0}{2} + \sqrt{\left(\frac{\omega_0}{2}\right)^2 + \omega_p^2} \end{cases}$$

$$\text{Evanesence} \rightarrow k \text{ imaginary} \rightarrow \begin{cases} \text{Mode (1)} & \omega_0 < \omega < \frac{\omega_0}{2} + \sqrt{\left(\frac{\omega_0}{2}\right)^2 + \omega_p^2} \\ \text{Mode (2)} & 0 < \omega < -\frac{\omega_0}{2} + \sqrt{\left(\frac{\omega_0}{2}\right)^2 + \omega_p^2} \end{cases}$$

$$e) \quad \frac{E_y}{E_x} = \frac{j\omega_p^2\omega_0}{\omega(\omega_0^2 - \omega^2) \left[ \frac{k_c^2 c^2}{\omega^2} - \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right) \right]} = \pm j \quad [\text{Circular polarization}]$$

$$f) \quad v_p = \frac{\omega}{k} = \frac{c}{\left[1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_0)}\right]}$$

$$E_0 e^{j(\omega t - k_0 z)} \bar{i}_x = \left\{ \frac{E_0}{2} [\bar{i}_x + j\bar{i}_y] + \frac{E_0}{2} [\bar{i}_x - j\bar{i}_y] \right\} e^{j(\omega t - k_0 z)}$$

$$g) \quad \bar{E}_T = \{ E_1 [\bar{i}_x + j\bar{i}_y] e^{j(\omega t - k_1 z)} + E_2 [\bar{i}_x - j\bar{i}_y] e^{j(\omega t - k_2 z)} \}$$

$$k_1 = \frac{\omega}{c} \left[ 1 - \frac{\omega_p^2}{\omega(\omega - \omega_0)} \right]$$

$$k_2 = \frac{\omega}{c} \left[ 1 - \frac{\omega_p^2}{\omega(\omega + \omega_0)} \right]$$

$$\hat{E}_T(z=0) = E_t \bar{i}_x = E_1(\bar{i}_x + j\bar{i}_y) + E_2(\bar{i}_x - j\bar{i}_y) \rightarrow E_1 = E_2 = \frac{E_t}{2}$$

$$\bar{E}_T = \frac{E_t}{2} \{ (\bar{i}_x + j\bar{i}_y)e^{-jk_1 z} + (\bar{i}_x - j\bar{i}_y)e^{-jk_2 z} \} e^{j\omega t}$$

$$\bar{H}_T = \frac{E_t}{2\omega\mu} \{ k_1(\bar{i}_x - j\bar{i}_y)e^{-jk_1 z} + k_2(\bar{i}_x + j\bar{i}_y)e^{-jk_2 z} \} e^{j\omega t}$$

$$17. \quad a) \quad \Delta\phi = \omega\ell \left( \frac{1}{c_{||}} - \frac{1}{c_{\perp}} \right) = \omega\ell\lambda B E_O^2 = 2\pi\ell c B E_O^2$$

$$b) \quad \Delta\phi = \frac{\pi}{2} = 2\pi\ell c B E_O^2 \rightarrow E_O^2 = \frac{1}{4B\ell c}$$

$$\Delta\phi = \pi = 2\pi\ell c B E_O^2 \rightarrow E_O^2 = \frac{1}{2B\ell c}$$

$$c) \quad \bar{E} = E_1 \text{Re } e^{j(\omega t - k_{||}\ell)} [\bar{i}_x + e^{j\phi} \bar{i}_y]; \quad \phi = (k_{||} - k_{\perp})\ell$$

Transmission axis of crossed polarizer in direction  $\bar{i}_x - \bar{i}_y$

$$E_t = \bar{E} \cdot \frac{(\bar{i}_x - \bar{i}_y)}{\sqrt{2}} = \frac{E_1}{\sqrt{2}} \text{Re } e^{j(\omega t - k_{||}\ell)} [1 - e^{j\phi}]$$

$$S_t = \frac{1}{2} \frac{|E_t|^2}{\eta}$$

$$= \frac{1}{4} \frac{E_1^2}{\eta} [(1 - \cos\phi)^2 + \sin^2\phi]$$

$$= \frac{1}{2} \frac{E_1^2}{\eta} (1 - \cos\phi)$$

$$= \frac{E_1^2}{\eta} \sin^2 \frac{\phi}{2}$$

$$\text{Max light} \rightarrow \phi = (2n + 1)\pi$$

$$n = 0, 1, 2, \dots$$

$$\text{Min light} \rightarrow \phi = 2n\pi$$

$$n = 1, 2, \dots$$

## Section 7.5

$$18. \text{ a) } \bar{E}_i = \text{Re} \hat{E}_i e^{-jk_o z} \bar{i}_y, \quad \bar{E}_r = \text{Re} \hat{E}_r e^{+jk_o z} \bar{i}_y, \quad \bar{E}_t = \text{Re} \hat{E}_t e^{-jk_t z} \bar{i}_y$$

$$\bar{H}_i = \text{Re} \frac{-\hat{E}_i}{\eta} e^{-jk_o z} \bar{i}_x, \quad \bar{H}_r = \text{Re} \frac{\hat{E}_r}{\eta} e^{+jk_o z} \bar{i}_x, \quad \bar{H}_t = \text{Re} \frac{-k_t}{\omega \mu} \hat{E}_t e^{-jk_t z} \bar{i}_x$$

$$k_o = \omega \sqrt{\epsilon_o \mu_o}, \quad k_t = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}$$

$$\text{b) } \hat{E}_i + \hat{E}_r = \hat{E}_t$$

$$\rightarrow \hat{E}_r = \frac{\hat{E}_i [1 - \frac{k_t \eta}{\omega \mu}]}{1 + \frac{k_t \eta}{\omega \mu}}, \quad \hat{E}_t = \frac{2\hat{E}_i}{1 + \frac{k_t \eta}{\omega \mu}}$$

$$-\hat{E}_i + \hat{E}_r = \frac{-k_t \eta}{\omega \mu} \hat{E}_t$$

$$\text{c) } \langle S \rangle_z = -\frac{1}{2} \text{Re} E_y H_x^*$$

$z > 0$  (Transmitted)

$$\langle S \rangle_z = \frac{1}{2} \text{Re} \left\{ |\hat{E}_t|^2 \frac{k_t^*}{\omega \mu} e^{-j(k_t - k_t^*)z} \right\}$$

$$= \begin{cases} \frac{1}{2} \frac{|\hat{E}_t|^2 k_t}{\omega \mu} = \frac{2|\hat{E}_i|^2 k_t}{\omega \mu [1 + \frac{k_t \eta}{\omega \mu}]^2} & \omega > \omega_p \text{ (} k_t \text{ real)} \\ 0 & \omega < \omega_p \text{ (} k_t \text{ imaginary)} \end{cases}$$

$z < 0$

$$\langle S \rangle_z = -\frac{1}{2\eta} \text{Re} [\hat{E}_i e^{-jk_o z} + \hat{E}_r e^{jk_o z}] [-\hat{E}_i^* e^{jk_o z} + \hat{E}_r^* e^{-jk_o z}]$$

$$= -\frac{1}{2\eta} \text{Re} \left\{ -|\hat{E}_i|^2 + |\hat{E}_r|^2 - \hat{E}_r \hat{E}_i^* e^{2jk_o z} + \hat{E}_i \hat{E}_r^* e^{-2jk_o z} \right\}$$

$$= \frac{1}{2\eta} [|\hat{E}_i|^2 - |\hat{E}_r|^2]$$

$$= \frac{|\hat{E}_i|^2}{2\eta} \left\{ 1 - \frac{\left| 1 - \frac{k_t \eta}{\omega \mu} \right|^2}{\left| 1 + \frac{k_t \eta}{\omega \mu} \right|^2} \right\}$$

$$k_t \text{ imaginary} \rightarrow \left| 1 - \frac{k_t \eta}{\omega \mu} \right| = \left| 1 + \frac{k_t \eta}{\omega \mu} \right| \rightarrow \langle S \rangle_z = 0$$

$$k_t \text{ real} \rightarrow \langle S \rangle_z = \frac{|\hat{E}_i|^2}{2\eta} \frac{4k_t \eta}{\omega \mu (1 + \frac{k_t \eta}{\omega \mu})^2} = \frac{2k_t |\hat{E}_i|^2}{\omega \mu [1 + \frac{k_t \eta}{\omega \mu}]^2}$$

19. a)

$$\bar{E} = E_x \cos(\omega t - kz) \bar{i}_x + E_y \sin(\omega t - kz) \bar{i}_y$$

$$\bar{H} = \frac{1}{\eta} [E_x \cos(\omega t - kz) \bar{i}_y - E_y \sin(\omega t - kz) \bar{i}_x]$$

$$\bar{S}_i = \bar{E} \times \bar{H} = \frac{1}{\eta} [E_x^2 \cos^2(\omega t - kz) + E_y^2 \sin^2(\omega t - kz)] \bar{i}_z$$

$$b) \quad \bar{E}_t = E_x \cos(\omega t - kz) \bar{i}_x$$

$$\bar{H}_t = \frac{E_x}{\eta} \cos(\omega t - kz) \bar{i}_y$$

$$\bar{S}_t = \frac{E_x^2}{\eta} \cos^2(\omega t - kz) \bar{i}_z$$

$$c) \quad |\bar{E}_{t2}| = |\bar{E}_t \cos \phi|$$

$$\rightarrow S_t = \frac{|\bar{E}_t|^2}{\eta} \cos^2 \phi \quad [\text{Law of Malus obeyed}]$$

$$|\bar{H}_{t2}| = \left| \frac{\bar{E}_t \cos \phi}{\eta} \right|$$

$$20. \quad a) \quad \bar{E}_i = \text{Re} E_o e^{j\omega(t - \frac{z}{c})} \bar{i}_y; \quad \bar{E}_r = \text{Re} \hat{E}_r e^{j\omega'(t + \frac{z}{c})}$$

$$\bar{H}_i = \text{Re} - \frac{E_o}{\eta} e^{j\omega(t - \frac{z}{c})} \bar{i}_x; \quad \bar{H}_r = \text{Re} \frac{\hat{E}_r}{\eta} e^{j\omega'(t + \frac{z}{c})}$$

$$b) \quad E_T(z=vt) = 0 \rightarrow E_o e^{j\omega t(1 - \frac{v}{c})} + \hat{E}_r e^{j\omega' t(1 + \frac{v}{c})} = 0$$

$$\omega(1 - \frac{v}{c}) = \omega'(1 + \frac{v}{c}) \rightarrow \omega' = \frac{\omega(1 - \frac{v}{c})}{1 + \frac{v}{c}} \approx \omega(1 - \frac{v}{c})^2 \approx \omega(1 - \frac{2v}{c})$$

$$\hat{E}_r = -E_o$$

$$\bar{E}_T = \bar{E}_i + \bar{E}_r = \text{Re} E_o [e^{j\omega(t - \frac{z}{c})} - e^{j\omega'(t + \frac{z}{c})}] \bar{i}_y = E_o [\cos\omega(t - \frac{z}{c}) - \cos\omega'(t + \frac{z}{c})] \bar{i}_y$$

$$\bar{H}_T = \bar{H}_i + \bar{H}_r = \text{Re} \frac{-E_o}{\eta} [e^{j\omega(t - \frac{z}{c})} + e^{j\omega'(t + \frac{z}{c})}] \bar{i}_x = \frac{-E_o}{\eta} [\cos\omega(t - \frac{z}{c}) + \cos\omega'(t + \frac{z}{c})] \bar{i}_x$$

$$c) \quad \bar{S} = \bar{E}_T \times \bar{H}_T = \frac{E_o^2}{\eta} [\cos^2\omega(t - \frac{z}{c}) - \cos^2\omega'(t + \frac{z}{c})] \bar{i}_z$$

We cannot use the complex Poynting vector because the incident and reflected waves are at different frequencies.

### Section 7.6

$$21. \quad a) \quad \bar{E}_i = \text{Re} \hat{E}_i e^{j(\omega t - k_1 z)} \bar{i}_x; \quad k_1 = \omega \sqrt{\epsilon_1 \mu_1}$$

$$\bar{H}_i = \text{Re} \frac{\hat{E}_i}{\eta_1} e^{j(\omega t - k_1 z)} \bar{i}_y; \quad \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\bar{E}_r = \text{Re} \hat{E}_r e^{j(\omega t + k_1 z)} \bar{i}_x$$

$$\bar{H}_r = \text{Re} -\frac{\hat{E}_r}{\eta} e^{j(\omega t + k_1 z)} \bar{i}_y$$

$$\bar{E}_t = \text{Re} \hat{E}_t \sin k_2(z - d) e^{j\omega t} \bar{i}_x; \quad k_2 = \omega \sqrt{\epsilon_2 \mu_2}, \quad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

$$\bar{H}_t = \text{Re} [-\frac{1}{j\omega\mu} \frac{\partial E_t}{\partial z}] \bar{i}_y = \text{Re} \frac{-k_2}{j\omega\mu} \hat{E}_t \cos k_2(z - d) e^{j\omega t} \bar{i}_y = \text{Re} \frac{j\hat{E}_t}{\eta_2} \cos k_2(z - d) e^{j\omega t} \bar{i}_y$$

b) Boundary Conditions:

$$\bar{E}_i(z=0) + \bar{E}_r(z=0) = \bar{E}_t(z=0) \rightarrow \hat{E}_i + \hat{E}_r = -\hat{E}_t \sin k_2 d$$

$$\bar{H}_i(z=0) + \bar{H}_r(z=0) = \bar{H}_t(z=0) \rightarrow \hat{E}_i - \hat{E}_r = \frac{j\eta_1}{\eta_2} \hat{E}_t \cos k_2 d$$

$$\hat{E}_r = \frac{-\hat{E}_i [j\eta_1 \cos k_2 d + \eta_2 \sin k_2 d]}{[j\eta_1 \cos k_2 d - \eta_2 \sin k_2 d]}, \quad \hat{E}_t = \frac{2\eta_2 \hat{E}_i}{j\eta_1 \cos k_2 d - \eta_2 \sin k_2 d}$$

$$c) \quad \langle \bar{S} \rangle = \frac{1}{2} \operatorname{Re} \hat{E} \times \hat{H}^*$$

$$z < 0$$

$$\langle S \rangle_z = \frac{1}{2\eta_1} \operatorname{Re} [\hat{E}_i e^{-jk_1 z} + \hat{E}_r e^{jk_1 z}] [\hat{E}_i^* e^{jk_1 z} - \hat{E}_r^* e^{-jk_1 z}]$$

$$= \frac{1}{2\eta_1} [|\hat{E}_i|^2 - |\hat{E}_r|^2]$$

$$= 0$$

$$z > 0$$

$$\langle S \rangle_z = \frac{1}{2\eta_2} \operatorname{Re} [\hat{E}_t \sin k_2 (z - d) (-j) \hat{E}_t^* \cos k_2 (z - d)]$$

$$= 0$$

$$d) \quad \hat{K}(z=d) = \hat{H}_y(z=d) \bar{i}_x = \frac{j\hat{E}_t}{\eta_2} \bar{i}_x$$

$$F_z = \frac{1}{4} \operatorname{Re} \mu_2 \hat{K}_x \hat{H}_y^*$$

$$= \frac{1}{4} \mu_2 |\hat{K}_x|^2$$

$$= \frac{1}{4} \mu_2 \frac{|\hat{E}_t|^2}{\eta_2^2} = \frac{1}{4} \epsilon_2 |\hat{E}_t|^2 = \frac{\epsilon_2 \eta_2^2 |\hat{E}_i|^2}{[\eta_1^2 \cos^2 k_2 d + \eta_2^2 \sin^2 k_2 d]}$$



## Section 7.7

$$22. \quad \vec{E} = \text{Re} \frac{\hat{E}}{e} j\omega t e^{-\vec{\gamma} \cdot \vec{r}} \quad -\vec{\gamma} \times \hat{E} = -j\omega\mu\hat{H}$$

$$\vec{H} = \text{Re} \frac{\hat{H}}{e} j\omega t e^{-\vec{\gamma} \cdot \vec{r}} \quad -\vec{\gamma} \times \hat{H} = j\omega\epsilon(1 + \frac{\sigma}{j\omega\epsilon})\hat{E}$$

$$-\vec{\gamma} \cdot \epsilon \hat{E} = 0, \quad -\vec{\gamma} \cdot \mu \hat{H} = 0$$

$$\vec{\gamma} \cdot \vec{\gamma} = (\alpha^2 - k^2 + 2j\alpha \cdot \vec{k}) = -\omega^2\mu\epsilon(1 + \frac{\sigma}{j\omega\epsilon})$$

$$\alpha^2 - k^2 = -\omega^2\mu\epsilon$$

$$\alpha \cdot \vec{k} = \frac{\sigma\omega\mu}{2} = \frac{1}{\delta^2}$$

$$23. \quad a) \quad \vec{H} = \begin{cases} \text{Re} \frac{\hat{H}_1}{e} j\omega t e^{-\vec{\gamma}_1 \cdot \vec{r}} \vec{i}_y \\ \text{Re} \frac{\hat{H}_2}{e} j\omega t e^{-\vec{\gamma}_2 \cdot \vec{r}} \vec{i}_y \end{cases}; \quad \vec{E} = \begin{cases} \text{Re} -\frac{\vec{\gamma}_1 \times \hat{H}_1 \vec{i}_y}{j\omega\epsilon} e^{j\omega t} e^{-\vec{\gamma}_1 \cdot \vec{r}} & z > 0 \\ \text{Re} -\frac{\vec{\gamma}_2 \times \hat{H}_2 \vec{i}_y}{j\omega\epsilon} e^{j\omega t} e^{-\vec{\gamma}_2 \cdot \vec{r}} & z < 0 \end{cases}$$

$$\vec{\gamma}_1 = jk_x \vec{i}_x + \gamma_z \vec{i}_z \quad \rightarrow \quad \gamma_z = [k_x^2 - \frac{\omega^2}{c^2}]^{1/2}$$

$$\vec{\gamma}_2 = jk_x \vec{i}_x - \gamma_z \vec{i}_z$$

$$\vec{\gamma}_1 \times \vec{H}_1 = [jk_x \vec{i}_x + \gamma_z \vec{i}_z] \times H_1 \vec{i}_y = H_1 [jk_x \vec{i}_z - \gamma_z \vec{i}_x]$$

$$\vec{\gamma}_2 \times \vec{H}_2 = H_2 [jk_x \vec{i}_z + \gamma_z \vec{i}_x]$$

$$\hat{E} = \begin{cases} -\frac{\hat{H}_1}{j\omega\epsilon} [jk_x \vec{i}_z - \gamma_z \vec{i}_x] e^{j\omega t} e^{-\vec{\gamma}_1 \cdot \vec{r}} & z > 0 \\ -\frac{\hat{H}_2}{j\omega\epsilon} [jk_x \vec{i}_z + \gamma_z \vec{i}_x] e^{j\omega t} e^{-\vec{\gamma}_2 \cdot \vec{r}} & z < 0 \end{cases}$$

$$E_x(z=0_+) = E_x(z=0_-) \rightarrow \hat{H}_1 = -\hat{H}_2$$

$$\epsilon(E_z(z=0_+) - E_z(z=0_-)) = \sigma_o \sin(\omega t - k_x x) \rightarrow \frac{k_x}{\omega} [-\hat{H}_1 + \hat{H}_2] = -j\sigma_o$$

$$\hat{H}_1 = -\hat{H}_2 = \frac{j\sigma_o \omega}{2k_x}$$

$$b) \quad \hat{K}_x(z=0) = -\hat{H}_y(z=0_+) + \hat{H}_y(z=0_-) = -\hat{H}_1 + \hat{H}_2 = -\frac{j\sigma_o \omega}{k_x}$$

$$\text{Check: } \nabla_{\Sigma} \cdot \vec{K} + \frac{\partial \sigma_f}{\partial t} = 0 \rightarrow \frac{d\hat{K}_x}{dx} + j\omega\hat{\sigma}_o = 0 \rightarrow \hat{K}_x = \frac{-j\omega\hat{\sigma}_o}{-jk_x} = \frac{\omega\hat{\sigma}_o}{k_x}$$

$$\hat{\sigma}_o = -j\sigma_o \rightarrow \hat{K}_x = \frac{-j\omega\sigma_o}{k_x}$$

$$24. a) \quad \gamma_{z1} = [k_x^2 - \frac{\omega^2}{c_o^2}]^{1/2}; \quad c_o = \frac{1}{\sqrt{\epsilon_o \mu_o}} \quad \text{Non-uniform} \rightarrow k_x > \frac{\omega}{c_o}$$

$$\rightarrow c < \frac{\omega}{k_x} < c_o$$

$$\gamma_{z2} = [k_x^2 - \frac{\omega^2}{c^2}]^{1/2}; \quad c = \frac{1}{\sqrt{\epsilon \mu}} \quad \text{Uniform} \rightarrow k_x < \frac{\omega}{c}$$

$$\text{Non-uniform in each region} \rightarrow k_x > \frac{\omega}{c} \rightarrow \omega < k_x c$$

$$\text{Uniform in each region} \rightarrow \omega > k_x c_o$$

$$b) \quad \hat{H}_y = \begin{cases} \hat{H}_1 e^{-jk_x x} e^{\gamma_{z1} z} & z < 0 \\ [\hat{H}_2 e^{-\gamma_{z1} z} + \hat{H}_3 e^{\gamma_{z1} z}] e^{-jk_x x} & 0 < z < d \\ \hat{H}_4 e^{-\gamma_{z2}(z-d)} e^{-jk_x x} & z > d \end{cases}$$

$$\hat{E} = \frac{\nabla \times \hat{H}}{j\omega\epsilon} = \begin{cases} -\frac{\hat{H}_1}{j\omega\epsilon_o} [jk_x \bar{i}_z + \gamma_{z1} \bar{i}_x] e^{-jk_x x} e^{\gamma_{z1} z} & z < 0 \\ -\frac{1}{j\omega\epsilon_o} \{ [jk_x \bar{i}_z - \gamma_{z1} \bar{i}_x] \hat{H}_2 e^{-\gamma_{z1} z} + [jk_x \bar{i}_z + \gamma_{z1} \bar{i}_x] \hat{H}_3 e^{\gamma_{z1} z} \} e^{-jk_x x} & 0 < z < d \\ -\frac{\hat{H}_4}{j\omega\epsilon} [jk_x \bar{i}_z - \gamma_{z2} \bar{i}_x] e^{-\gamma_{z2}(z-d)} e^{-jk_x x} & z > d \end{cases}$$

Boundary Conditions:

$$H_y(z=0_-) - H_y(z=0_+) = K_o \rightarrow \hat{H}_1 - \hat{H}_2 - \hat{H}_3 = K_o$$

$$E_x(z=0_-) = E_x(z=0_+) \rightarrow -\hat{H}_1 = \hat{H}_2 - \hat{H}_3$$

$$H_y(z=d_-) = H_y(z=d_+) \rightarrow \hat{H}_2 e^{-\gamma_{z1} d} + \hat{H}_3 e^{\gamma_{z1} d} = \hat{H}_4$$

$$E_x(z=d_-) = E_x(z=d_+) \rightarrow -\hat{H}_2 e^{-\gamma_{z1} d} + \hat{H}_3 e^{\gamma_{z1} d} = -\frac{\gamma_{z2}}{\gamma_{z1}} \frac{\epsilon_o}{\epsilon} \hat{H}_4$$

$$\hat{H}_1 = \frac{K_o}{2} \left[ \frac{1 + \frac{\epsilon_o}{\epsilon} \frac{\gamma_{z2}}{\gamma_{z1}} - (1 - \frac{\epsilon_o}{\epsilon} \frac{\gamma_{z2}}{\gamma_{z1}}) e^{-2\gamma_{z1} d}}{1 + \frac{\epsilon_o}{\epsilon} \frac{\gamma_{z2}}{\gamma_{z1}}} \right]; \quad \hat{H}_2 = -\frac{K_o}{2}$$

$$\hat{H}_3 = \frac{-\frac{K_o}{2} e^{-2\gamma_{z1} d} (1 - \frac{\epsilon_o}{\epsilon} \frac{\gamma_{z2}}{\gamma_{z1}})}{1 + \frac{\epsilon_o}{\epsilon} \frac{\gamma_{z2}}{\gamma_{z1}}}; \quad \hat{H}_4 = \frac{-K_o e^{-\gamma_{z1} d}}{1 + \frac{\epsilon_o}{\epsilon} \frac{\gamma_{z2}}{\gamma_{z1}}}$$

c)  $\gamma_{z1}$  real (non-uniform),  $\gamma_{z2}$  imaginary (uniform)

$z < 0$

$$\begin{aligned} \langle \bar{S} \rangle &= \frac{1}{2} \operatorname{Re} \bar{E} \times \bar{H}^* = \frac{1}{2} \operatorname{Re} \left[ \frac{-\hat{H}_1}{j\omega\epsilon_o} [jk_x \bar{i}_z + \gamma_{z1} \bar{i}_x] e^{\gamma_{z1} z} \times \hat{H}_1^* e^{\gamma_{z1}^* z} \bar{i}_y \right] \\ &= -\frac{|\hat{H}_1|^2}{2} \operatorname{Re} \left\{ \frac{-jk_x \bar{i}_x + \gamma_{z1} \bar{i}_z}{j\omega\epsilon_o} e^{(\gamma_{z1} + \gamma_{z1}^*) z} \right\} \\ &= \frac{|\hat{H}_1|^2 k_x}{2\omega\epsilon_o} e^{2\gamma_{z1} z} \bar{i}_x \end{aligned}$$

$$\langle S_z \rangle = 0$$

$z > d$

$$\begin{aligned} \langle \bar{S} \rangle &= \frac{1}{2} \operatorname{Re} \left[ \frac{-\hat{H}_4}{j\omega\epsilon} [jk_x \bar{i}_z - \gamma_{z2} \bar{i}_x] e^{-\gamma_{z2}(z-d)} \times \hat{H}_4^* e^{-\gamma_{z2}^*(z-d)} \bar{i}_y \right] \\ &= -\frac{1}{2} |\hat{H}_4|^2 \operatorname{Re} \left[ \frac{-jk_x \bar{i}_x - \gamma_{z2} \bar{i}_z}{j\omega\epsilon} e^{-(z-d)(\gamma_{z2} + \gamma_{z2}^*)} \right] \\ &= \frac{1}{2} |\hat{H}_4|^2 \frac{k_x \bar{i}_x + |\gamma_{z2}| \bar{i}_z}{\omega\epsilon} \end{aligned}$$

$$\begin{aligned}
&= \frac{K_o^2 e^{-2\gamma_{z1}d} \left[ k_x \bar{i}_x + |\gamma_{z2}| \bar{i}_z \right]}{2 \left[ 1 + \left( \frac{\epsilon_o}{\epsilon} \frac{|\gamma_{z2}|}{\gamma_{z1}} \right)^2 \right] \omega \epsilon} \\
\langle S_z \rangle &= \frac{K_o^2 e^{-2\gamma_{z1}d} |\gamma_{z2}|}{2\omega \epsilon \left[ 1 + \left( \frac{\epsilon_o}{\epsilon} \frac{|\gamma_{z2}|}{\gamma_{z1}} \right)^2 \right]} \\
0 < z < d \\
\langle \bar{S} \rangle &= \frac{1}{2} \operatorname{Re} \left\{ \frac{\hat{H}_2 e^{-\gamma_{z1}z} (jk_x \bar{i}_z - \gamma_{z1} \bar{i}_x) + \hat{H}_3 e^{\gamma_{z1}z} (jk_x \bar{i}_z + \gamma_{z1} \bar{i}_x)}{j\omega \epsilon_o} \times [\hat{H}_2^* e^{-\gamma_{z1}z} + \hat{H}_3^* e^{\gamma_{z1}z}] \bar{i}_y \right\} \\
&= \frac{1}{2\omega \epsilon_o} \operatorname{Re} -j \{ |\hat{H}_2|^2 e^{-2\gamma_{z1}z} (-jk_x \bar{i}_x - \gamma_{z1} \bar{i}_z) + |\hat{H}_3|^2 e^{2\gamma_{z1}z} (-jk_x \bar{i}_x + \gamma_{z1} \bar{i}_z) \\
&\quad + \hat{H}_2 \hat{H}_3^* (-jk_x \bar{i}_x - \gamma_{z1} \bar{i}_z) + \hat{H}_3 \hat{H}_2^* (-jk_x \bar{i}_x + \gamma_{z1} \bar{i}_z) \} \\
&= \frac{1}{2\omega \epsilon_o} \{ -k_x [|\hat{H}_2|^2 e^{-2\gamma_{z1}z} + |\hat{H}_3|^2 e^{2\gamma_{z1}z} + 2\operatorname{Re}(\hat{H}_2 \hat{H}_3^*)] \bar{i}_x + 2\gamma_{z1} \operatorname{Re} \hat{H}_2 \hat{H}_3^* \bar{i}_z \} \\
\langle S_z \rangle &= \frac{\gamma_{z1}}{\omega \epsilon_o} \operatorname{Re} j \hat{H}_2 \hat{H}_3^* \\
&= \frac{\gamma_{z1}}{\omega \epsilon_o} \left( \frac{K_o}{2} \right)^2 e^{-2\gamma_{z1}d} \frac{2\epsilon_o}{\epsilon} \frac{|\gamma_{z2}|}{\gamma_{z1}} = \frac{K_o^2 e^{-2\gamma_{z1}d} |\gamma_{z2}|}{2\omega \epsilon \left[ 1 + \left( \frac{\epsilon_o}{\epsilon} \frac{|\gamma_{z2}|}{\gamma_{z1}} \right)^2 \right]} \\
\langle S_z \rangle \Big|_{0 < z < d} &= \langle S_z \rangle \Big|_{z > d}
\end{aligned}$$

## Section 7.8

25. a)  $\bar{E}_i = \operatorname{Re} \hat{E}_i e^{j(\omega t - k_{xi}x - k_{zi}z)} \bar{i}_y$ ;  $k_{xi} = k_o \sin \theta$

$\bar{H}_i = \operatorname{Re} \frac{\hat{E}_i}{\eta} [-\cos \theta \bar{i}_x + \sin \theta \bar{i}_z] e^{j(\omega t - k_{xi}x - k_{zi}z)}$ ;  $k_{zi} = k_o \cos \theta$

$$\bar{E}_r = \text{Re} \hat{E}_r e^{j(\omega t + k_{xi}x + k_{zi}z)} \bar{i}_y ; \quad \eta = \sqrt{\frac{\mu}{\epsilon}}$$

$$\bar{H}_r = \text{Re} \frac{\hat{E}_r}{\eta} [\cos \theta \bar{i}_x - \sin \theta \bar{i}_z] e^{j(\omega t + k_{xi}x + k_{zi}z)}$$

$$\bar{E}_1 = \text{Re} \hat{E}_1 e^{j(\omega t - k_{xi}x + k_{zi}z)} \bar{i}_y$$

$$\bar{H}_1 = \text{Re} \frac{\hat{E}_1}{\eta} [\cos \theta \bar{i}_x + \sin \theta \bar{i}_z] e^{j(\omega t - k_{xi}x + k_{zi}z)}$$

$$\bar{E}_2 = \text{Re} \hat{E}_2 e^{j(\omega t + k_{xi}x - k_{zi}z)} \bar{i}_y$$

$$\bar{H}_2 = \text{Re} \frac{\hat{E}_2}{\eta} [-\cos \theta \bar{i}_x - \sin \theta \bar{i}_z] e^{j(\omega t + k_{xi}x - k_{zi}z)}$$

$$b) \quad \bar{E}_T(z=0) = 0 \rightarrow (\hat{E}_i + \hat{E}_1) e^{-jk_{xi}x} + (\hat{E}_r + \hat{E}_2) e^{jk_{xi}x} = 0$$

$$\hat{E}_i + \hat{E}_1 = 0, \quad \hat{E}_r + \hat{E}_2 = 0$$

$$\bar{E}_T(x=0) = 0 \rightarrow (\hat{E}_i + \hat{E}_2) e^{-jk_{zi}z} + (\hat{E}_r + \hat{E}_1) e^{jk_{zi}z} = 0$$

$$\hat{E}_i + \hat{E}_2 = 0, \quad \hat{E}_r + \hat{E}_1 = 0 \rightarrow \hat{E}_1 = -\hat{E}_i, \quad \hat{E}_2 = -\hat{E}_i, \quad \hat{E}_r = \hat{E}_i$$

$$\hat{E}_T(x, z) = \hat{E}_i \{ e^{-j(k_{xi}x + k_{zi}z)} + e^{j(k_{xi}x + k_{zi}z)} - e^{-j(k_{xi}x - k_{zi}z)} - e^{j(k_{xi}x - k_{zi}z)} \} \bar{i}_y$$

$$= 2\hat{E}_i \{ \cos(k_{xi}x + k_{zi}z) - \cos(k_{xi}x - k_{zi}z) \} \bar{i}_y$$

$$= -4\hat{E}_i \sin k_{xi}x \sin k_{zi}z \bar{i}_y$$

$$\bar{E}_T(x, z, t) = \text{Re} \hat{E}_T(x, z) e^{j\omega t} = -4E_o \sin k_{xi}x \sin k_{zi}z \cos \omega t \bar{i}_y$$

$$\begin{aligned} \hat{H}_T(x, z) = \frac{\hat{E}_i}{\eta} \{ & -\cos \theta \bar{i}_x [e^{-j(k_{xi}x + k_{zi}z)} - e^{j(k_{xi}x + k_{zi}z)} - e^{-j(k_{xi}x - k_{zi}z)} + e^{j(k_{xi}x - k_{zi}z)}] \\ & + \sin \theta \bar{i}_z [e^{-j(k_{xi}x + k_{zi}z)} - e^{j(k_{xi}x + k_{zi}z)} - e^{-j(k_{xi}x - k_{zi}z)} + e^{j(k_{xi}x - k_{zi}z)}] \} \end{aligned}$$

$$\begin{aligned}
&= 2j \frac{\hat{E}_i}{\eta} \left\{ \cos\theta \bar{i}_x [\sin(k_{xi}x + k_{zi}z) + \sin(k_{xi}x - k_{zi}z)] \right. \\
&\quad \left. - \sin\theta \bar{i}_z [\sin(k_{xi}x + k_{zi}z) - \sin(k_{xi}x - k_{zi}z)] \right\} \\
&= 4j \frac{\hat{E}_i}{\eta} \left\{ \cos\theta \sin k_{xi}x \cos k_{zi}z \bar{i}_x - \sin\theta \cos k_{xi}x \sin k_{zi}z \bar{i}_z \right\}
\end{aligned}$$

$$\bar{H}_T(x, z, t) = \text{Re} \hat{H}_T(x, z) e^{j\omega t} = \frac{-4E_o}{\eta} \left\{ \cos\theta \sin k_{xi}x \cos k_{zi}z \bar{i}_x - \sin\theta \cos k_{xi}x \sin k_{zi}z \bar{i}_z \right\} \sin\omega t$$

$$c) \quad \hat{K}(z=0) = -\hat{H}_x(z=0) \bar{i}_y = \frac{-4j\hat{E}_i}{\eta} \cos\theta \sin k_{xi}x$$

$$\hat{K}(x=0) = \hat{H}_z(x=0) \bar{i}_y = \frac{-4j\hat{E}_i}{\eta} \sin\theta \sin k_{zi}z$$

$$\hat{\sigma}_f(x=0) = \hat{\sigma}_f(z=0) = 0$$

$$d) \quad F_z(z=0) = \frac{1}{2} \mu_o |\hat{K}(z=0)|^2 = \frac{8\mu_o}{\eta} |\hat{E}_i|^2 \cos^2\theta \sin^2 k_{xi}x = 8\epsilon_o |\hat{E}_i|^2 \cos^2\theta \sin^2 k_{xi}x$$

$$F_x(z=0) = \frac{1}{2} \mu_o |\hat{K}(x=0)|^2 = 8\epsilon_o |\hat{E}_i|^2 \sin^2\theta \sin^2 k_{zi}z$$

$$e) \quad \bar{S} = \bar{E} \times \bar{H}$$

$$= \frac{-16E_o^2}{\eta} \sin k_{xi}x \sin k_{zi}z \sin\omega t \cos\omega t \left\{ \cos\theta \sin k_{xi}x \cos k_{zi}z \bar{i}_z + \sin\theta \cos k_{xi}x \sin k_{zi}z \bar{i}_x \right\}$$

### Section 7.9

$$26. \quad a) \quad \cos\theta_i = \frac{h_1}{s_i} \rightarrow s_i = \frac{h_1}{\cos\theta_i} \quad \rightarrow \quad t = \frac{s_i + s_r}{c_1} = \frac{1}{c_1} \left[ \frac{h_1}{\cos\theta_i} + \frac{h_2}{\cos\theta_r} \right]$$

$$\cos\theta_r = \frac{h_2}{s_r} \rightarrow s_r = \frac{h_2}{\cos\theta_r}$$

$$\sin\theta_i = \frac{L_1}{s_i}, \quad \sin\theta_r = \frac{L_2}{s_r} \rightarrow L_1 + L_2 = L_{AB} = s_i \sin\theta_i + s_r \sin\theta_r$$

$$= h_1 \tan\theta_i + h_2 \tan\theta_r$$

$$b) \quad 0 = \frac{h_1}{\cos^2 \theta_i} d\theta_i + \frac{h_2}{\cos^2 \theta_r} d\theta_r \rightarrow \frac{d\theta_r}{d\theta_i} = -\frac{h_1}{h_2} \frac{\cos^2 \theta_r}{\cos^2 \theta_i}$$

$$dt = 0 = \frac{1}{c_1} \left[ \frac{h_1 \sin \theta_i d\theta_i}{\cos^2 \theta_i} + \frac{h_2 \sin \theta_r d\theta_r}{\cos^2 \theta_r} \right] \rightarrow \frac{d\theta_r}{d\theta_i} = \frac{-h_1}{h_2} \frac{\sin \theta_i}{\sin \theta_r} \frac{\cos^2 \theta_r}{\cos^2 \theta_i}$$

$$\rightarrow \sin \theta_i = \sin \theta_r$$

$$c) \quad t = \frac{1}{c_1} \frac{h_1}{\cos \theta_i} + \frac{1}{c_2} \frac{h_2}{\cos \theta_t}$$

$$L_{AC} = h_1 \tan \theta_i + h_2 \tan \theta_t$$

$$d) \quad 0 = \frac{h_1}{\cos^2 \theta_i} d\theta_i + \frac{h_2}{\cos^2 \theta_t} d\theta_t \rightarrow \frac{d\theta_t}{d\theta_i} = \frac{-h_1}{h_2} \frac{\cos^2 \theta_t}{\cos^2 \theta_i}$$

$$dt = 0 = \frac{h_1}{c_1 \cos^2 \theta_i} \sin \theta_i d\theta_i + \frac{h_2}{c_2 \cos^2 \theta_t} \sin \theta_t d\theta_t \rightarrow \frac{d\theta_t}{d\theta_i} = \frac{-h_1 c_2}{h_2 c_1} \frac{\sin \theta_i \cos^2 \theta_t}{\sin \theta_t \cos^2 \theta_i}$$

$$\rightarrow c_2 \sin \theta_i = c_1 \sin \theta_t \quad [\text{Snell's law}]$$

$$27. \quad \mu = \mu_o, \quad \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_o}{\epsilon_1}}; \quad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_o}{\epsilon_2}}$$

$$\underline{\vec{E}} \parallel \text{interface} \quad (\sin \theta_t = \frac{c_2}{c_1} \sin \theta_i = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin \theta_i)$$

$$R = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = \left( \frac{\cos \theta_i}{\sqrt{\epsilon_2}} - \frac{\cos \theta_t}{\sqrt{\epsilon_1}} \right) / \left( \frac{\cos \theta_i}{\sqrt{\epsilon_2}} + \frac{\cos \theta_t}{\sqrt{\epsilon_1}} \right)$$

$$= \frac{\sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin \theta_i \cos \theta_i - \sin \theta_i \cos \theta_t}{\sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin \theta_i \cos \theta_i + \sin \theta_i \cos \theta_t}$$

$$\begin{aligned}
&= \frac{\sin\theta_t \cos\theta_i - \sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t} \\
&= \frac{-\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}
\end{aligned}$$

$$\begin{aligned}
T &= \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{\frac{2\cos\theta_i}{\sqrt{\epsilon_2}}}{\frac{\cos\theta_i}{\sqrt{\epsilon_2}} + \frac{\cos\theta_t}{\sqrt{\epsilon_1}}} = \frac{2\sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin\theta_i \cos\theta_i}{\sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin\theta_i \cos\theta_i + \sin\theta_i \cos\theta_t} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}
\end{aligned}$$

$\vec{H} \parallel \text{interface}$

$$\begin{aligned}
R &= \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \left( \frac{\cos\theta_i}{\sqrt{\epsilon_1}} - \frac{\cos\theta_t}{\sqrt{\epsilon_2}} \right) / \left( \frac{\cos\theta_t}{\sqrt{\epsilon_2}} + \frac{\cos\theta_i}{\sqrt{\epsilon_1}} \right) \\
&= \frac{\sqrt{\frac{\epsilon_2}{\epsilon_1}} \sin\theta_t \cos\theta_i - \sin\theta_t \cos\theta_t}{\sin\theta_t \cos\theta_t + \sqrt{\frac{\epsilon_2}{\epsilon_1}} \sin\theta_t \cos\theta_i} \\
&= \frac{\sin\theta_i \cos\theta_i - \sin\theta_t \cos\theta_t}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i} \\
&= \frac{\sin(\theta_i - \theta_t) \cos(\theta_i + \theta_t)}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} \\
&= \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}
\end{aligned}$$



$$\begin{aligned}
T &= \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \frac{\frac{2\cos\theta_i}{\sqrt{\epsilon_2}}}{\frac{\cos\theta_t}{\sqrt{\epsilon_2}} + \frac{\cos\theta_i}{\sqrt{\epsilon_1}}} \\
&= \frac{2\cos\theta_i \sin\theta_t}{\sin\theta_t \cos\theta_t + \sqrt{\frac{\epsilon_2}{\epsilon_1}} \cos\theta_i \sin\theta_t} \\
&= \frac{2\cos\theta_i \sin\theta_t}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i} \\
&= \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_t + \theta_i) \cos(\theta_i - \theta_t)}
\end{aligned}$$

28.  $n = A + \frac{B}{\lambda^2}$ ;  $A = 1.5$ ,  $B = 5 \times 10^{-15}$

Color	$\lambda$ (meters)	$n$	$\theta_t$ [ $\sin\theta_t = \frac{\sin\theta_i}{n}$ ]
violet	$4 \times 10^{-7}$	1.5313	$19.06^\circ$
blue	$4.5 \times 10^{-7}$	1.5247	$19.14^\circ$
green	$5.5 \times 10^{-7}$	1.5165	$19.25^\circ$
yellow	$6 \times 10^{-7}$	1.5139	$19.29^\circ$
orange	$6.5 \times 10^{-7}$	1.5118	$19.31^\circ$
red	$7 \times 10^{-7}$	1.5102	$19.33^\circ$

29. a)  $\bar{E}_i = \text{Re} \hat{E}_i [\cos\theta_i \bar{i}_x - \sin\theta_i \bar{i}_z] e^{j(\omega t - k_{xi}x - k_{zi}z)}$ ;  $k_{xi} = k_1 \sin\theta_i$ ;  $k_1 = \omega \sqrt{\epsilon_1 \mu_1}$

$$\bar{H}_i = \text{Re} \frac{\hat{E}_i}{\eta_1} e^{j(\omega t - k_{xi}x - k_{zi}z)} \bar{i}_y; \quad k_{zi} = k_1 \cos\theta_i, \quad \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\bar{E}_r = \text{Re} \hat{E}_r [-\cos\theta_i \bar{i}_x - \sin\theta_i \bar{i}_z] e^{j(\omega t - k_{xi}x + k_{zi}z)}$$

$$\bar{H}_r = \text{Re} \frac{\hat{E}_r}{\eta_1} e^{j(\omega t - k_{xi}x + k_{zi}z)} \bar{i}_y$$

$$\bar{E}_1 = \text{Re} \hat{E}_1 [\cos \theta_t \bar{i}_x - \sin \theta_t \bar{i}_z] e^{j(\omega t - k_{xt}x - k_{zt}z)}; \quad k_{xt} = k_2 \sin \theta_t; \quad k_2 = \omega \sqrt{\epsilon_2 \mu_2}$$

$$\bar{H}_1 = \text{Re} \frac{\hat{E}_1}{\eta_2} e^{j(\omega t - k_{xt}x - k_{zt}z)}; \quad k_{zt} = k_2 \cos \theta_t; \quad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

$$\bar{E}_2 = \text{Re} \hat{E}_2 [-\cos \theta_t \bar{i}_x - \sin \theta_t \bar{i}_z] e^{j(\omega t - k_{xt}x + k_{zt}z)}$$

$$\bar{H}_2 = \text{Re} \frac{\hat{E}_2}{\eta_2} e^{j(\omega t - k_{xt}x + k_{zt}z)} \bar{i}_y$$

$$\bar{E}_t = \text{Re} \hat{E}_t [\cos \theta_i \bar{i}_x - \sin \theta_i \bar{i}_z] e^{j(\omega t - k_{xi}x - k_{zi}(z-d))}$$

$$\bar{H}_t = \text{Re} \frac{\hat{E}_t}{\eta_1} e^{j(\omega t - k_{xi}x - k_{zi}(z-d))} \bar{i}_y$$

Boundary Conditions:

$$H_y(z=0_-) = H_y(z=0_+) \rightarrow \frac{\hat{E}_i + \hat{E}_r}{\eta_1} = \frac{\hat{E}_1 + \hat{E}_2}{\eta_2}$$

$$H_y(z=d_-) = H_y(z=d_+) \rightarrow \frac{\hat{E}_1 e^{-jk_{zt}d} + \hat{E}_2 e^{jk_{zt}d}}{\eta_2} = \frac{\hat{E}_t}{\eta_1}$$

$$E_x(z=0_-) = E_x(z=0_+) \rightarrow (\hat{E}_i - \hat{E}_r) \cos \theta_i = (\hat{E}_1 - \hat{E}_2) \cos \theta_t$$

$$E_x(z=d_-) = E_x(z=d_+) \rightarrow (\hat{E}_1 e^{-jk_{zt}d} - \hat{E}_2 e^{jk_{zt}d}) \cos \theta_t = \hat{E}_t \cos \theta_i$$

$$k_1 \sin \theta_i = k_2 \sin \theta_t \rightarrow \sqrt{\epsilon_1 \mu_1} \sin \theta_i = \sqrt{\epsilon_2 \mu_2} \sin \theta_t$$

$$\begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ 1 & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix} \begin{bmatrix} \hat{E}_r \\ \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_t \end{bmatrix} = \begin{bmatrix} \hat{E}_i \\ 0 \\ \hat{E}_i \\ 0 \end{bmatrix}$$

$$D = \det \begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ 1 & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix}$$

$$= -4\cos k_{zt}d - 2js\sin k_{zt}d \left[ \frac{\eta_1}{\eta_2} \frac{\cos\theta_i}{\cos\theta_t} + \frac{\eta_2}{\eta_1} \frac{\cos\theta_t}{\cos\theta_i} \right]$$

$$\hat{E}_r = \frac{1}{D} \det \begin{bmatrix} \hat{E}_i & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ \hat{E}_i & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix}$$

$$= \frac{2j\hat{E}_i}{D} \sin k_{zt} d \left[ \frac{\eta_2}{\eta_1} \frac{\cos \theta_t}{\cos \theta_i} - \frac{\eta_1}{\eta_2} \frac{\cos \theta_i}{\cos \theta_t} \right]$$

$$\hat{E}_1 = \frac{1}{D} \det \begin{bmatrix} -1 & \hat{E}_i & \frac{\eta_1}{\eta_2} & 0 \\ 0 & 0 & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ 1 & \hat{E}_i & -\frac{\cos \theta_t}{\cos \theta_i} & 0 \\ 0 & 0 & -e^{jk_{zt}d} & -\frac{\cos \theta_i}{\cos \theta_t} \end{bmatrix} = \frac{-2\hat{E}_i e^{jk_{zt}d}}{D} \left[ \frac{\cos \theta_i}{\cos \theta_t} + \frac{\eta_2}{\eta_1} \right]$$

$$\hat{E}_2 = \frac{1}{D} \det \begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \hat{E}_i & 0 \\ 0 & e^{-jk_{zt}d} & 0 & -\frac{\eta_2}{\eta_1} \\ 1 & \frac{\cos \theta_t}{\cos \theta_i} & \hat{E}_i & 0 \\ 0 & e^{-jk_{zt}d} & 0 & -\frac{\cos \theta_i}{\cos \theta_t} \end{bmatrix} = \frac{2\hat{E}_i e^{-jk_{zt}d}}{D} \left[ \frac{\cos \theta_i}{\cos \theta_t} - \frac{\eta_2}{\eta_1} \right]$$

$$\hat{E}_t = \frac{1}{D} \det \begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & \hat{E}_i \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & 0 \\ 1 & \frac{\cos \theta_t}{\cos \theta_i} & -\frac{\cos \theta_t}{\cos \theta_i} & \hat{E}_i \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & 0 \end{bmatrix} = \frac{-4\hat{E}_i}{D}$$

Check: Special Case,  $n_1 = n_2$ ,  $\theta_t = \theta_i$

$$D = -4e^{jk_{zt}d}$$

$$\hat{E}_r = 0, \hat{E}_1 = 1, \hat{E}_2 = 0, \hat{E}_t = \hat{E}_i e^{-jk_{zt}d}$$

b) Uniform plane waves in middle region  $\rightarrow k_{zt}$  real  $\rightarrow \cos\theta_t$  real

$$\rightarrow \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}} \sin\theta_i < 1 \rightarrow \sin\theta_i < \sqrt{\frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}}$$

$$c) \langle \bar{S} \rangle_t = \frac{1}{2} \operatorname{Re} \hat{E}_t \times \hat{H}_t^*$$

$$= \frac{1}{2} \operatorname{Re} \frac{|\hat{E}_t|^2}{n_1} [\cos\theta_i \bar{i}_z + \sin\theta_i \bar{i}_x]$$

Uniform plane wave in middle region  $[\cos\theta_t \text{ real}]$

$$|\hat{E}_t|^2 = \frac{4|\hat{E}_i|^2}{4\cos^2 k_{zt}d + \sin^2 k_{zt}d \left[ \frac{n_1}{n_2} \frac{\cos\theta_i}{\cos\theta_t} + \frac{n_2}{n_1} \frac{\cos\theta_t}{\cos\theta_i} \right]^2}$$

Non-uniform plane wave in middle region  $[\cos\theta_t = -j|\cos\theta_t|]$

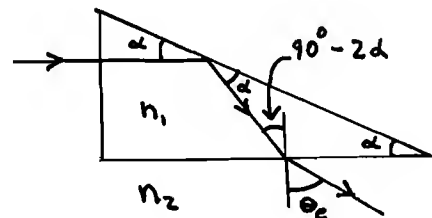
$$|\hat{E}_t|^2 = \frac{4|\hat{E}_i|^2}{\{2\cos k_{zt}d - \sin k_{zt}d \left[ \frac{n_1}{n_2} \frac{\cos\theta_i}{|\cos\theta_t|} + \frac{n_2}{n_1} \frac{|\cos\theta_t|}{\cos\theta_i} \right]\}^2}$$

Oppositely traveling non-uniform plane waves in middle region allow time average power flow although each non-uniform plane wave alone cannot carry time average power.

### Section 7.10

$$30. \quad a) \text{ and } b) \quad \sin\theta_c = \frac{n_2}{n_1}, \quad \sin\theta_e = \frac{n_1}{n_2} \cos 2\alpha = \frac{\cos 2\alpha}{\sin\theta_c}$$

$$\sin\theta_e = \frac{n_1}{n_2} \sin(90^\circ - 2\alpha)$$



# ELECTRODYNAMICS - FIELDS AND WAVES

$\alpha$	$\theta_c = 90^\circ - \alpha$	$n_2$	$n_1$	$\theta_e$
$45^\circ$	$45^\circ$	1	1.414	0
		1.33	1.88	
$30^\circ$	$60^\circ$	1	1.155	$35.27^\circ$
		1.33	1.536	
$60^\circ$	$30^\circ$	1	2	$-90^\circ$
		1.33	2.66	

$$\begin{aligned}
 31. \quad \sin\theta_t &= 1.33 \sin\theta_i \\
 &= (1.33)(.5) = .665 \rightarrow \theta_t = 41.69^\circ
 \end{aligned}$$

$$32. \quad a) \quad \sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1$$

$$\sin\theta_3 = \frac{n_2}{n_3} \sin\theta_2 = \frac{n_1}{n_3} \sin\theta_1$$

$$b) \quad \sin\theta_{1c} = \frac{n_3}{n_1}$$

c) Same critical angle with or without coating.

$$d) \quad \sin\theta_1 > \frac{n_2}{n_1} \quad [\text{Coating must have smaller index of refraction than light pipe.}]$$

$$33. \quad a) \quad \sin\theta_t = \frac{1}{n} \sin\theta = \frac{x}{nR}$$

$$\frac{y - R}{\sin\theta_t} = \frac{R}{\sin(\theta - \theta_t)} \rightarrow y = R \left[ \frac{\sin\theta_t}{\sin(\theta - \theta_t)} + 1 \right]$$

$$\sin(\theta - \theta_t) = \sin\theta \cos\theta_t - \sin\theta_t \cos\theta$$

$$= \sin\theta \sqrt{1 - \sin^2\theta_t} - \sin\theta_t \sqrt{1 - \sin^2\theta}$$

$$= \frac{x}{R} \sqrt{1 - \left(\frac{x}{nR}\right)^2} - \frac{x}{nR} \sqrt{1 - \left(\frac{x}{R}\right)^2}$$

$$y = R \left[ \frac{1}{\sqrt{n^2 - \left(\frac{x}{R}\right)^2} - \sqrt{1 - \left(\frac{x}{R}\right)^2}} + 1 \right]$$

$$y < 2R \rightarrow \sqrt{n^2 - \left(\frac{x}{R}\right)^2} - \sqrt{1 - \left(\frac{x}{R}\right)^2} > 1$$

$$n^2 - \left(\frac{x}{R}\right)^2 > 1 + \left(1 - \left(\frac{x}{R}\right)^2\right) + 2\sqrt{1 - \left(\frac{x}{R}\right)^2}$$

$$\sqrt{1 - \left(\frac{x}{R}\right)^2} < \frac{n^2 - 2}{2}$$

$$n^2 > 2 \rightarrow 1 - \left(\frac{x}{R}\right)^2 < \frac{(n^2 - 2)^2}{4} \rightarrow \left(\frac{x}{R}\right)^2 > 1 - \frac{(n^2 - 2)^2}{4} \quad ; \quad \sqrt{2} \leq n \leq 2$$

$$b) \quad \sin\theta_t = \frac{\sin\theta}{n} = \frac{\alpha}{n}$$

$$\frac{\sin\theta_t}{R'} = \frac{\sin(90^\circ + \theta - \theta_t)}{R} = \frac{\cos(\theta - \theta_t)}{R}$$

$$\begin{aligned} R' &= \frac{R \sin\theta_t}{\cos(\theta - \theta_t)} = \frac{R\alpha}{n[\cos\theta\cos\theta_t + \sin\theta\sin\theta_t]} \\ &= \frac{\alpha R}{n[\sqrt{1 - \alpha^2} \sqrt{1 - \left(\frac{\alpha}{n}\right)^2} + \frac{\alpha^2}{n}]} \\ &= \frac{\alpha R}{[\sqrt{n^2 - (n\alpha)^2} \sqrt{n^2 - \alpha^2} + \alpha^2]} \end{aligned}$$

## CHAPTER 8

## GUIDED ELECTROMAGNETIC WAVES

Section 8.1

1. From (2.6.4c)

$$C = \begin{cases} \frac{2\pi\epsilon_o}{\cosh^{-1}(\frac{s}{a} + 1)} & \text{wire-plane} \\ \frac{2\pi\epsilon_o}{\cosh^{-1}[\frac{D^2}{2a^2} - 1]} & \text{2 wire line} \end{cases}$$

$$L = \frac{\epsilon_o \mu_o}{C} = \begin{cases} \frac{\mu_o}{2\pi} \cosh^{-1}(\frac{s}{a} + 1) & \text{wire-plane} \\ \frac{\mu_o}{2\pi} \cosh^{-1}[\frac{D^2}{2a^2} - 1] & \text{2 wire line} \end{cases}$$

$$2. \quad a) \quad \frac{\partial v}{\partial z} = -L(z) \frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -C(z) \frac{\partial v}{\partial t}$$

$$\frac{\partial^2 v}{\partial z \partial t} = -L(z) \frac{\partial^2 i}{\partial t^2} = -\frac{\partial}{\partial z} \left[ \frac{1}{C(z)} \frac{\partial i}{\partial z} \right] \rightarrow \frac{\partial^2 i}{\partial t^2} = \frac{1}{L(z)} \frac{\partial}{\partial z} \left[ \frac{1}{C(z)} \frac{\partial i}{\partial z} \right]$$

$$\frac{\partial^2 i}{\partial z \partial t} = -C(z) \frac{\partial^2 v}{\partial t^2} = -\frac{\partial}{\partial z} \left[ \frac{1}{L(z)} \frac{\partial v}{\partial z} \right] \rightarrow \frac{\partial^2 v}{\partial t^2} = \frac{1}{C(z)} \frac{\partial}{\partial z} \left[ \frac{1}{L(z)} \frac{\partial v}{\partial z} \right]$$

$$b) \quad L(z) = L_o e^{\alpha z}, \quad C(z) = C_o e^{-\alpha z}$$

$$v(z, t) = \text{Re} \hat{v}(z) e^{j\omega t}, \quad i(z, t) = \text{Re} \hat{i}(z) e^{j\omega t}$$



$$\frac{\partial^2 i}{\partial t^2} - \frac{1}{L(z)C(z)} \frac{\partial^2 i}{\partial z^2} + \frac{1}{L(z)C^2(z)} \frac{\partial i}{\partial z} \frac{dC(z)}{dz} = 0$$

$$\frac{\partial^2 v}{\partial t^2} - \frac{1}{L(z)C(z)} \frac{\partial^2 v}{\partial z^2} + \frac{1}{L^2(z)C(z)} \frac{\partial v}{\partial z} \frac{dL(z)}{dz} = 0$$

$$\frac{\partial^2 i}{\partial t^2} - \frac{1}{L_o C_o} \frac{\partial^2 i}{\partial z^2} - \frac{\alpha}{L_o C_o} \frac{\partial i}{\partial z} = 0$$

$$\frac{\partial^2 v}{\partial t^2} - \frac{1}{L_o C_o} \frac{\partial^2 v}{\partial z^2} + \frac{\alpha}{L_o C_o} \frac{\partial v}{\partial z} = 0$$

$$\frac{d^2 \hat{i}(z)}{dz^2} + \alpha \frac{d\hat{i}(z)}{dz} + L_o C_o \omega^2 \hat{i}(z) = 0$$

$$\hat{i}(z) = \hat{i} e^{pz} \rightarrow p^2 + \alpha p + L_o C_o \omega^2 = 0$$

$$p = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - L_o C_o \omega^2} = -\frac{\alpha}{2} \pm j\beta; \quad \beta = \sqrt{L_o C_o \omega^2 - \frac{\alpha^2}{4}}$$

$$\hat{i}(z) = [I_1 e^{j\beta z} + I_2 e^{-j\beta z}] e^{-\frac{\alpha}{2} z}$$

$$\hat{v}(z) = -\frac{1}{j\omega C(z)} \frac{d\hat{i}}{dz} = -\frac{1}{j\omega C_o} [I_1 (j\beta - \frac{\alpha}{2}) e^{j\beta z} - I_2 (j\beta + \frac{\alpha}{2}) e^{-j\beta z}] e^{\frac{\alpha}{2} z}$$

$$c) \quad \hat{v}(z=0) = V_o = -\frac{1}{j\omega C_o} [I_1 (j\beta - \frac{\alpha}{2}) - I_2 (j\beta + \frac{\alpha}{2})]$$

$$\text{Open Circuited End: } \hat{i}(z=l) = 0 \rightarrow I_1 e^{j\beta l} + I_2 e^{-j\beta l} = 0$$

$$I_1 = \frac{-j\omega C_o V_o e^{-j\beta l}}{[(j\beta + \frac{\alpha}{2}) e^{j\beta l} + (j\beta - \frac{\alpha}{2}) e^{-j\beta l}]}; \quad I_2 = \frac{j\omega C_o V_o e^{j\beta l}}{[(j\beta + \frac{\alpha}{2}) e^{j\beta l} + (j\beta - \frac{\alpha}{2}) e^{-j\beta l}]}$$

$$\begin{aligned} \hat{i}(z) &= \frac{-j\omega C_o V_o}{[(j\beta + \frac{\alpha}{2}) e^{j\beta l} + (j\beta - \frac{\alpha}{2}) e^{-j\beta l}]} [e^{j\beta(z-l)} - e^{-j\beta(z-l)}] e^{-\frac{\alpha}{2} z} \\ &= \frac{2\omega C_o V_o \sin\beta(z-l) e^{-\frac{\alpha}{2} z}}{[(j\beta + \frac{\alpha}{2}) e^{j\beta l} + (j\beta - \frac{\alpha}{2}) e^{-j\beta l}]} \end{aligned}$$

$$\hat{v}(z) = -\frac{1}{j\omega C(z)} \frac{d\hat{i}}{dz} = \frac{2jV_o [\beta \cos\beta(z-l) - \frac{\alpha}{2} \sin\beta(z-l)] e^{\frac{\alpha}{2}z}}{[(j\beta + \frac{\alpha}{2})e^{j\beta l} + (j\beta - \frac{\alpha}{2})e^{-j\beta l}]}$$

$$\text{Short Circuited End: } \hat{v}(z=l) = 0 \rightarrow I_1(j\beta - \frac{\alpha}{2})e^{j\beta l} - I_2(j\beta + \frac{\alpha}{2})e^{-j\beta l} = 0$$

$$I_1 = \frac{\omega C_o V_o e^{-j\beta l}}{2(j\beta - \frac{\alpha}{2})\sin\beta l}, \quad I_2 = \frac{\omega C_o V_o e^{j\beta l}}{2(j\beta + \frac{\alpha}{2})\sin\beta l}$$

$$\begin{aligned} \hat{v}(z) &= \frac{jV_o}{2\sin\beta l} [e^{j\beta(z-l)} - e^{-j\beta(z-l)}] e^{\frac{\alpha}{2}z} \\ &= -\frac{V_o \sin\beta(z-l) e^{\frac{\alpha}{2}z}}{\sin\beta l} \end{aligned}$$

$$\hat{i}(z) = -\frac{1}{j\omega L(z)} \frac{d\hat{v}}{dz} = \frac{V_o}{jL_o \omega \sin\beta l} [\beta \cos\beta(z-l) + \frac{\alpha}{2} \sin\beta(z-l)] e^{-\frac{\alpha}{2}z}$$

$$d) \text{ Waves decay if } \frac{\alpha^2}{4} > \omega^2 L_o C_o$$

$$\text{Cut off } \rightarrow \omega_{co} = \frac{\alpha}{2\sqrt{L_o C_o}}$$

$$e) \text{ Short circuited line resonance } \rightarrow \sin\beta l = 0 \rightarrow \beta l = l \sqrt{L_o C_o \omega^2 - \frac{\alpha^2}{4}} = n\pi \quad n=1,2,3\dots$$

$$\omega_n = \left[ \frac{(\frac{n\pi}{l})^2 + \frac{\alpha^2}{4}}{L_o C_o} \right]^{1/2}$$

$$f) \text{ Open circuited line resonance } \rightarrow j\beta(e^{j\beta l} + e^{-j\beta l}) + \frac{\alpha}{2}(e^{j\beta l} - e^{-j\beta l}) = 0$$

$$\beta \cos\beta l + \frac{\alpha}{2} \sin\beta l = 0 \rightarrow \tan\beta l = -\frac{2\beta}{\alpha}$$

$$3. a) \quad \bar{E} = \frac{v(t)}{\alpha r} \bar{i}_\phi, \quad \sigma_f = \epsilon E_\phi = \frac{\epsilon v(t)}{\alpha r}, \quad q_t = \int_a^b \sigma_f D dr = \frac{\epsilon v(t) D}{\alpha} \ln \frac{b}{a}$$

$$\bar{H} = H_r(r) \bar{i}_r \rightarrow \Phi = \mu H_r(r) r \alpha D \rightarrow H_r(r) = \frac{\Phi}{\mu \alpha D r}$$

$$\int_a^b H_r(r) dr = \frac{\Phi}{\mu \alpha D} \ln \frac{b}{a} = i \rightarrow H_r(r) = \frac{i}{r \ln \frac{b}{a}}$$

$$b) \quad C = \frac{q_T}{Dv(t)} = \frac{\epsilon \ell_n \frac{b}{a}}{\alpha}, \quad L = \frac{\Phi}{Di} = \frac{\mu \alpha}{\ell_n \frac{b}{a}}$$

$$LC = \epsilon \mu, \quad Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}} \frac{\alpha}{\ell_n \frac{b}{a}}$$

$$4. \quad a) \quad \nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}_f + \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow -\frac{\partial H_y}{\partial z} = J_x + \epsilon \frac{\partial E_x}{\partial t}$$

$$-\frac{\partial^2 H_y}{\partial z \partial t} = \frac{\partial J_x}{\partial t} + \epsilon \frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\mu} \frac{\partial^2 E_x}{\partial z^2} \rightarrow \omega_p^2 E_x + \frac{\partial^2 E_x}{\partial t^2} - c^2 \frac{\partial^2 E_x}{\partial z^2} = 0$$

$$b) \quad E_x(z, t) = \frac{v(z, t)}{s} \rightarrow \frac{\partial^2 v}{\partial t^2} + \omega_p^2 v - c^2 \frac{\partial^2 v}{\partial z^2} = 0$$

$$H_y(z, t) = \frac{i(z, t)}{D}$$

$$c) \quad v(z, t) = \text{Re} \hat{v} e^{j(\omega t - kz)} \rightarrow -\omega^2 + \omega_p^2 + k^2 c^2 = 0$$

$$k = \pm \frac{\sqrt{\omega^2 - \omega_p^2}}{c}$$

$$k \text{ real (propagation)} \rightarrow \omega > \omega_p$$

$$k \text{ imaginary (evanescence)} \quad \omega < \omega_p$$

$$d) \quad \hat{v}(z) = V_1 \sin kz + V_2 \cos kz$$

$$\hat{v}(z=0) = 0 = V_2$$

$$\hat{v}(z=-\ell) = V_o = -V_1 \sin k\ell$$

$$\hat{v}(z) = -\frac{V_o \sin kz}{\sin k\ell}, \quad v(z, t) = \text{Re} \hat{v}(z) e^{j\omega t} = -\frac{V_o \sin kz}{\sin k\ell} \cos \omega t$$

$$-\frac{\mu}{D} \frac{\partial i}{\partial t} = \frac{1}{s} \frac{\partial v}{\partial z} \rightarrow \hat{i}(z) = + \frac{D V_o k}{\mu s j \omega} \frac{\cos kz}{\sin k\ell}$$

$$i(z, t) = \operatorname{Re} \hat{i}(z) e^{j\omega t} = \frac{V_o k D}{\omega \mu s} \frac{\cos kz}{\sin k\ell} \sin \omega t$$

$$e) \text{ Resonance } \rightarrow \sin k\ell = 0 \rightarrow k\ell = n\pi \rightarrow \omega^2 = \omega_p^2 + \left(\frac{n\pi c}{\ell}\right)^2$$

$$5. \quad a) \quad i(z, t) = \frac{C}{\Delta z} \frac{\partial}{\partial t} [v(z - \Delta z) - v(z)]; \quad v(z, t) = \frac{L}{\Delta z} \frac{\partial}{\partial t} [i(z) - i(z + \Delta z)]$$

$$\lim_{z \rightarrow 0} i(z, t) = -C \frac{\partial^2 v}{\partial t \partial z}; \quad v(z, t) = -L \frac{\partial^2 i}{\partial t \partial z}$$

$$b) \quad i(z, t) = \operatorname{Re} \hat{i} e^{j(\omega t - kz)}, \quad v(z, t) = \operatorname{Re} \hat{v} e^{j(\omega t - kz)}$$

$$\hat{i} = -C\omega k \hat{v}; \quad \hat{v} = -L\omega k \hat{i}$$

$$\hat{i} = +LC\omega^2 k^2 \hat{i} \rightarrow LC\omega^2 k^2 = 1 \rightarrow k = \frac{1}{\omega \sqrt{LC}}$$

$$c) \quad v_p = \frac{\omega}{k} = \omega^2 \sqrt{LC}$$

$$v_g = \frac{d\omega}{dk} = -\omega^2 \sqrt{LC}$$

Such systems are called backward wave because the group velocity is opposite in direction to the phase velocity.

$$d) \quad \hat{v}(z) = V_1 \sin kz + V_2 \cos kz$$

$$\hat{v}(z=0) = 0 = V_2$$

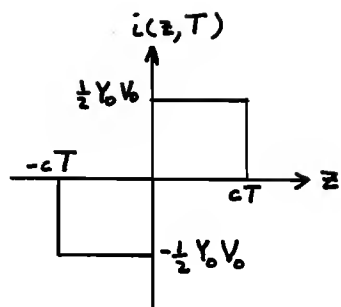
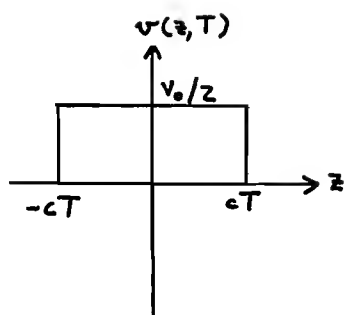
$$\hat{v}(z=-\ell) = V_o = -V_1 \sin k\ell \rightarrow \hat{v}(z) = \frac{-V_o}{\sin k\ell} \sin kz$$

$$\hat{i}(z) = -Cj\omega \frac{d\hat{v}}{dz} = \frac{j\omega C V_o k \cos kz}{\sin k\ell} = j \sqrt{\frac{C}{L}} V_o \frac{\cos kz}{\sin k\ell}$$

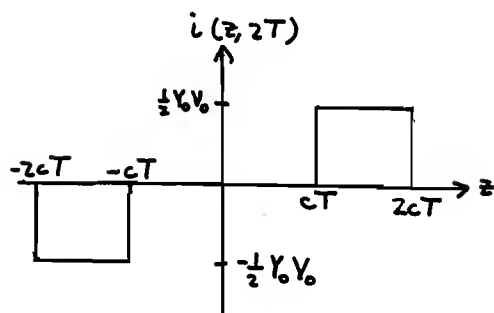
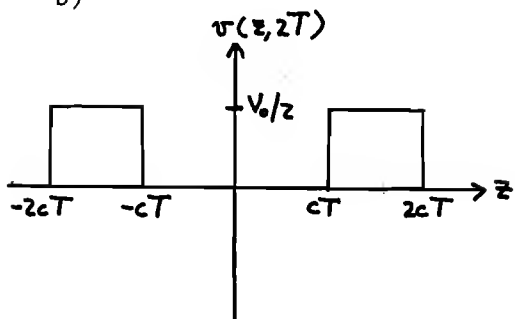
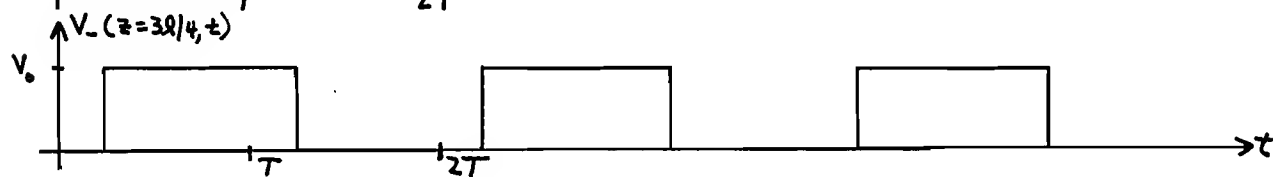
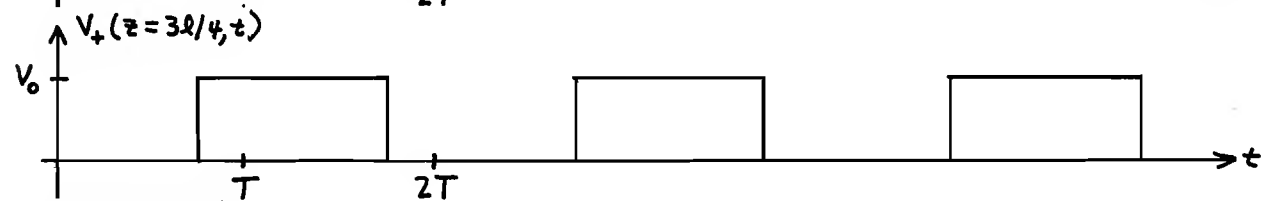
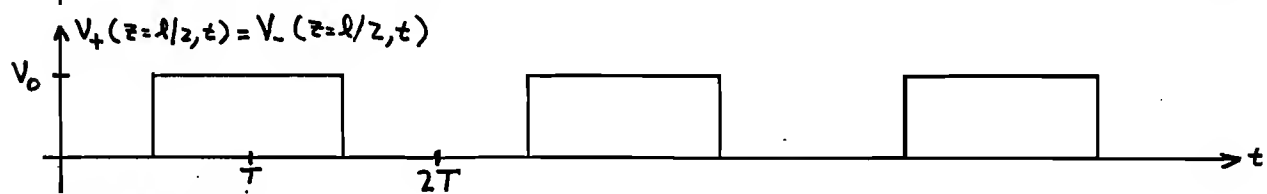
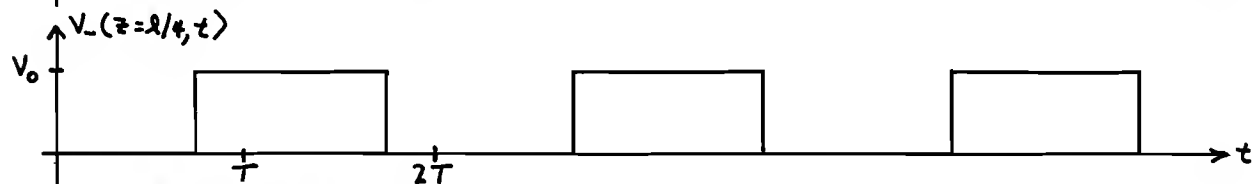
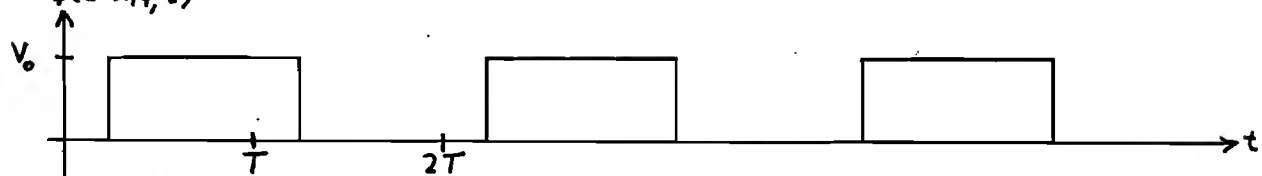
$$e) \text{ Resonance } \rightarrow \sin k\ell = 0 \rightarrow k\ell = n\pi \rightarrow \omega_n = \frac{1}{\left(\frac{n\pi}{\ell}\right) \sqrt{LC}}$$

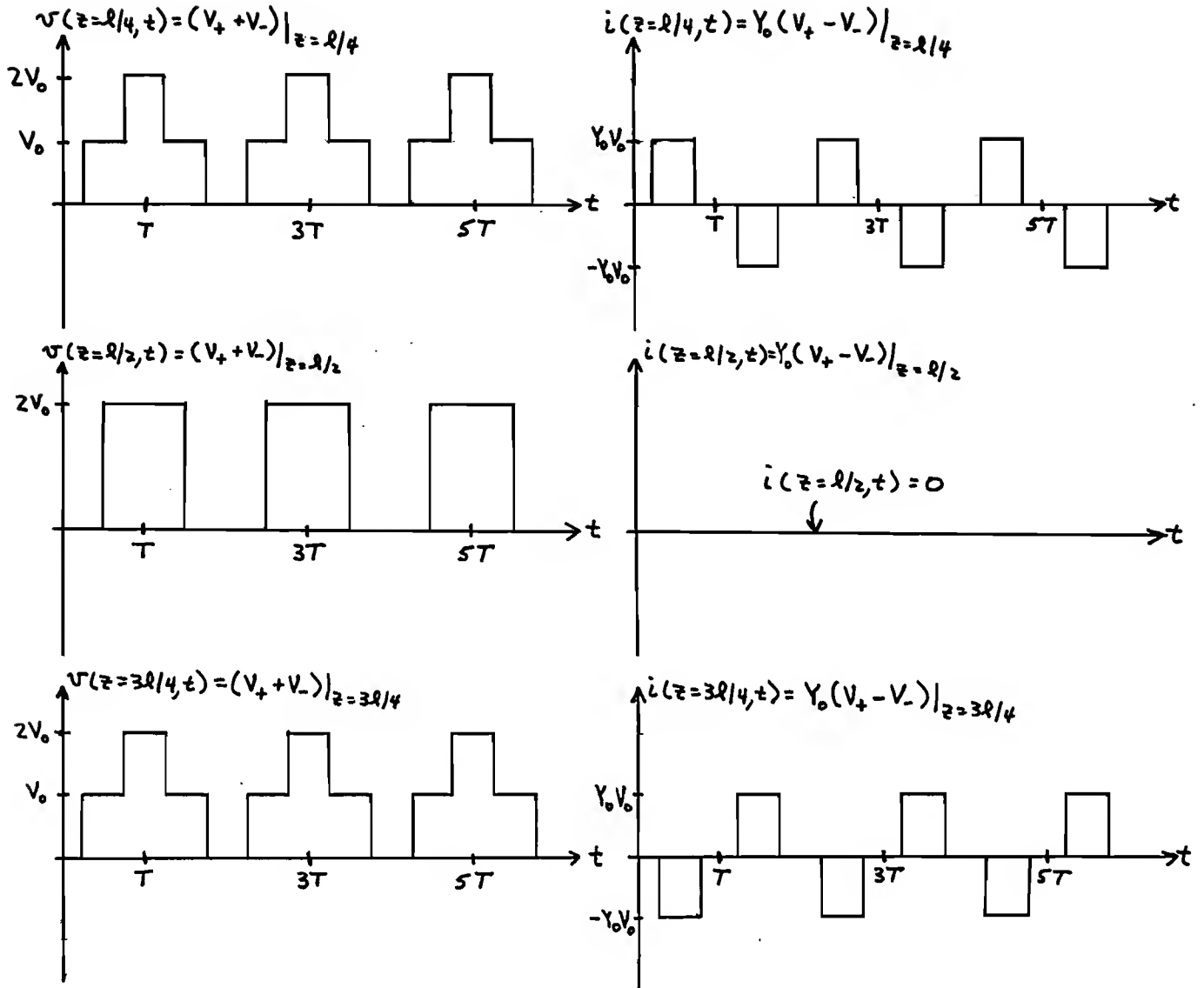
## Section 8.2

6. a)



b)


 7.  $V_+(z=l/4, t)$ 




$$8. \quad a) \quad v(t=0) = \frac{V_o R_L}{R_L + R_s} = V_+ + V_- \quad V_+ = \frac{V_o}{2} \frac{(R_L + Z_o)}{(R_L + R_s)}$$

$$i(t=0) = \frac{V_o}{R_L + R_s} = Y_o(V_+ - V_-) \quad \rightarrow \quad V_- = \frac{V_o}{2} \frac{(R_L - Z_o)}{(R_L + R_s)}$$

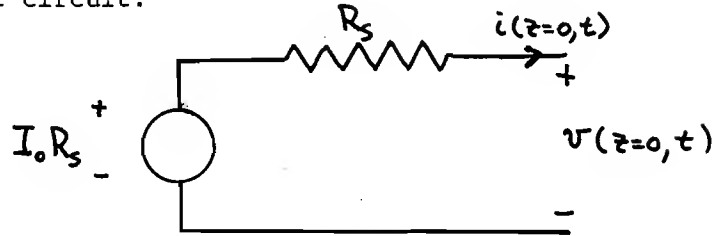
$$b) \quad V_{+n} = A(\Gamma_s \Gamma_L)^n; \quad \Gamma_s = \frac{R_s - Z_o}{R_s + Z_o}, \quad \Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

$$V_{-n} = \Gamma_L V_{+n} = A \Gamma_L (\Gamma_s \Gamma_L)^n;$$

$$V_{+n=0} = A = \frac{V_o}{2} \left( \frac{R_L + Z_o}{R_L + R_s} \right)$$

$$\begin{aligned} V_n &= V_{+n} + V_{-n} = \frac{V_o}{2} \left( \frac{R_L + Z_o}{R_L + R_s} \right) \left[ 1 + \frac{R_L - Z_o}{R_L + Z_o} \right] (\Gamma_s \Gamma_L)^n \\ &= \frac{V_o R_L}{R_L + R_s} (\Gamma_s \Gamma_L)^n \end{aligned}$$

9. Thevenin equivalent circuit:



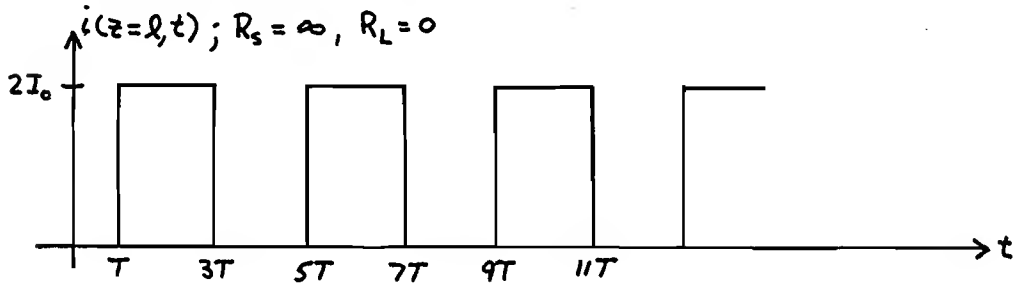
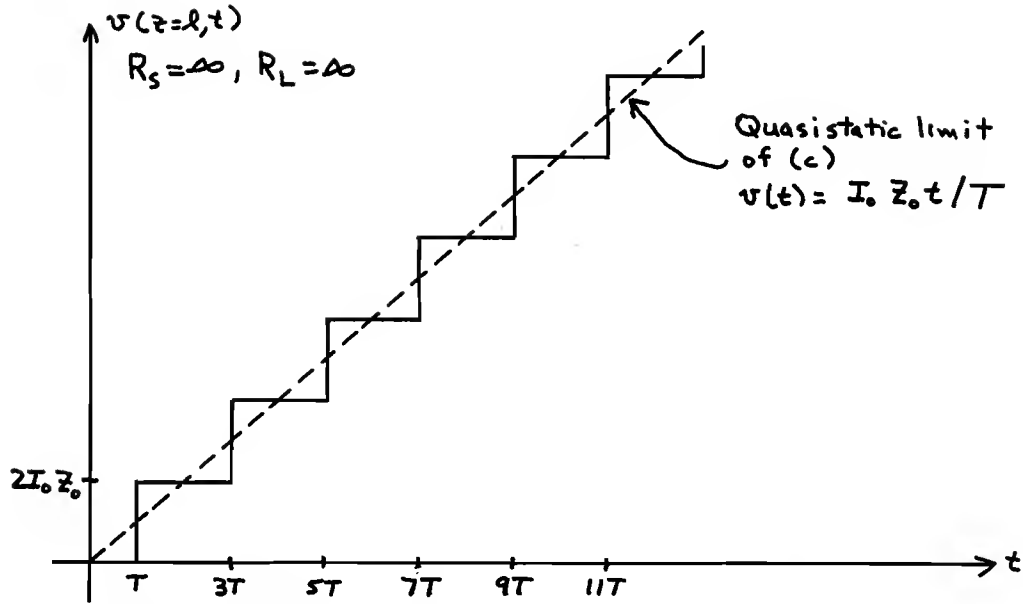
$$a) \quad V_n = \frac{I_o R_L R_s}{R_L + R_s} [1 - (\Gamma_s \Gamma_L)^n]; \quad \Gamma_s = \frac{R_s - Z_o}{R_s + Z_o}, \quad \Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

$$I_n = \frac{V_n}{R_L} = \frac{I_o R_s}{R_L + R_s} [1 - (\Gamma_s \Gamma_L)^n]$$

$$b) \quad V_n = \frac{I_o R_s \Gamma_o (1 + \Gamma_L) [1 - (\Gamma_s \Gamma_L)^n]}{1 - \Gamma_s \Gamma_L}; \quad \Gamma_o = \frac{Z_o}{R_s + Z_o}$$

$$\lim_{\substack{R_L \rightarrow \infty \\ R_s \rightarrow \infty}} V_n = \lim_{\substack{\Gamma_s \rightarrow 1 \\ \Gamma_L \rightarrow 1}} V_n = 2 I_o Z_o$$

$$\lim_{\substack{R_L \rightarrow 0 \\ R_s \rightarrow \infty}} I_n = \lim_{\substack{\Gamma_s \rightarrow 1 \\ \Gamma_L \rightarrow -1}} I_n = I_o [1 - (-1)^n] = \begin{cases} 0 & n \text{ even} \\ 2 I_o & n \text{ odd} \end{cases}$$

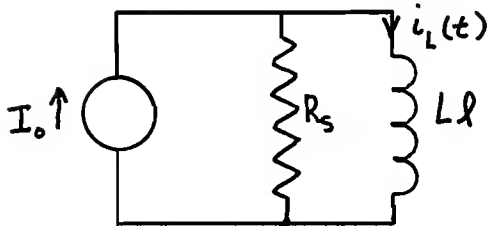


c)  $I_0 = C\ell \frac{dv}{dt} \rightarrow V = \frac{I_0}{C\ell} t = I_0 Z_0 \frac{t}{T}; T = \ell\sqrt{LC}, Z_0 = \sqrt{\frac{L}{C}}$

d)  $R_L = 0 \rightarrow \Gamma_L = -1$

$$I_n = I_0 [1 - (\Gamma_s)^n]; \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} = \frac{R_s - \sqrt{\frac{L}{C}}}{R_s + \sqrt{\frac{L}{C}}}$$

e)



$$i_L(t) = I_0 (1 - e^{-t/\tau}); \tau = \frac{L\ell}{R}$$

$$\Gamma_s = \frac{\frac{R_s}{L\ell} - \frac{1}{\ell\sqrt{LC}}}{\frac{R_s}{L\ell} + \frac{1}{\ell\sqrt{LC}}} = \frac{\frac{1}{\tau} - \frac{1}{T}}{\frac{1}{\tau} + \frac{1}{T}} = \frac{T - \tau}{T + \tau}$$

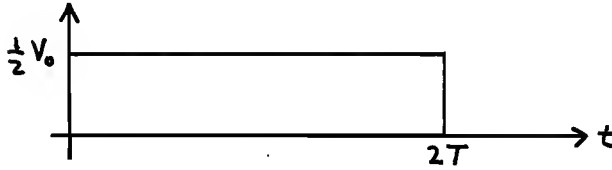
d) and e) approximately equal if  $\frac{T}{\tau} \ll 1$  ( $T = \ell\sqrt{LC}$ )



$$10. \quad a) \quad v(t=0) = V_o = V_+ + V_- \rightarrow V_+ = V_- = \frac{V_o}{2}$$

$$i(t=0) = 0 = Y_o (V_+ - V_-)$$

$$b) \quad v(z=l, t) = V_+(z=l, t); \quad V_-(z=l, t) = 0$$

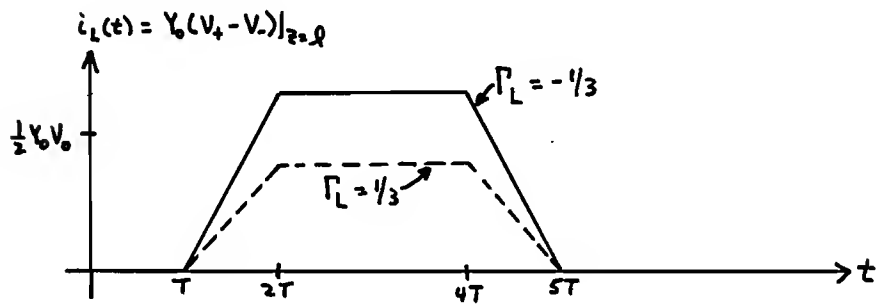
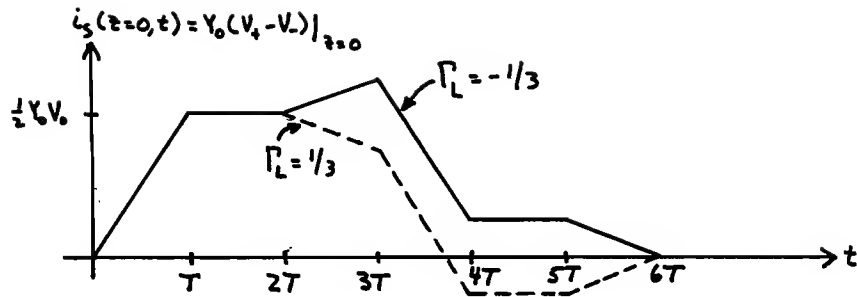


$$11. \quad \Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

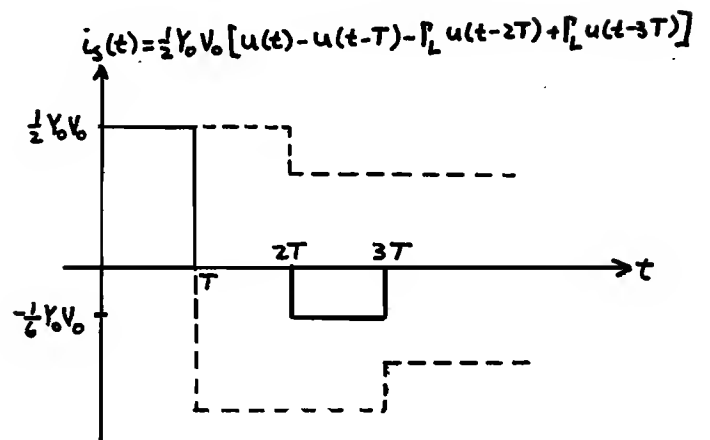
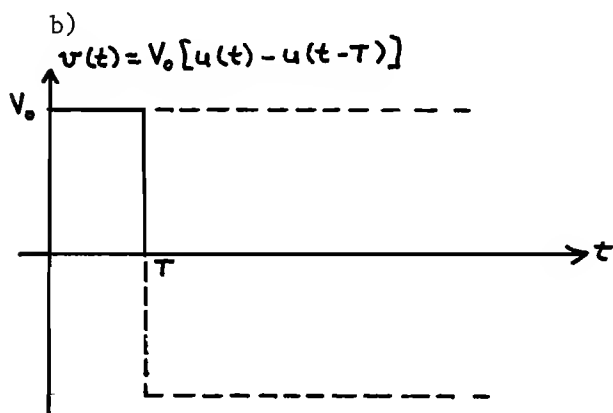
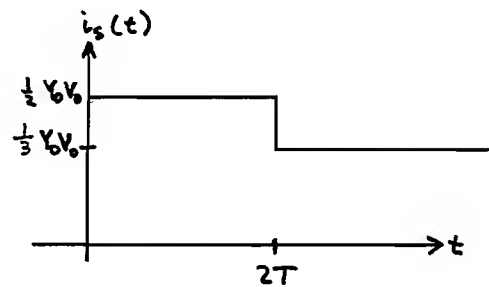
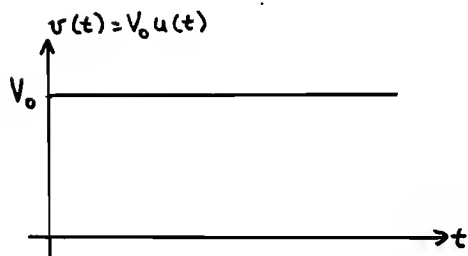
$$R_L = 2Z_o \rightarrow \Gamma_L = \frac{1}{3}$$

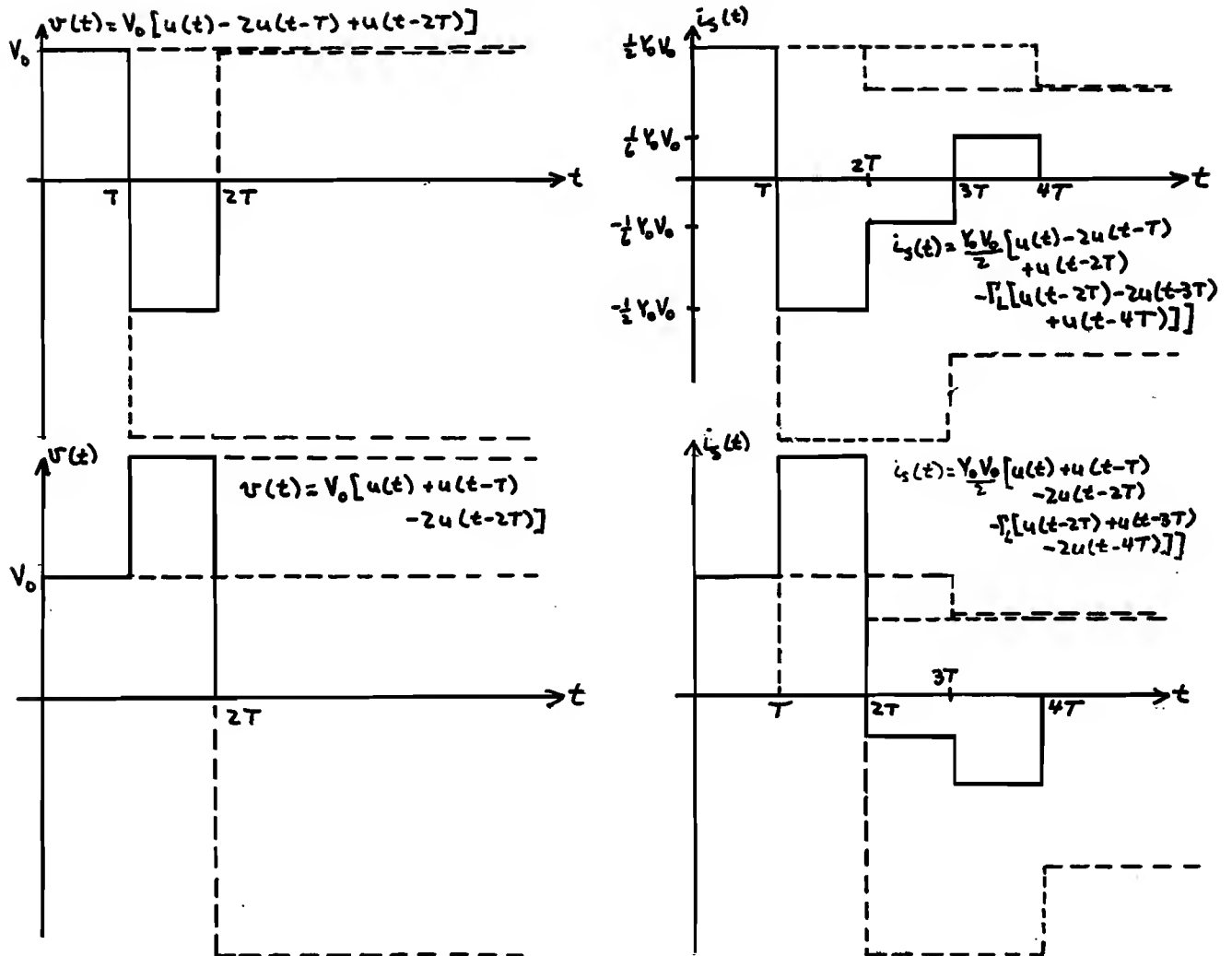
$$R_L = \frac{1}{2} Z_o \rightarrow \Gamma_L = -\frac{1}{3}$$

t	z = 0	z = l
0 < t < T	$V_+ = \frac{V_o t}{2T}, \quad V_- = 0$	$V_+ = 0, \quad V_- = 0$
T < t < 2T	$V_+ = \frac{V_o}{2}, \quad V_- = 0$	$V_+ = \frac{V_o (t-T)}{2T}, \quad V_- = \Gamma_L V_+$
2T < t < 3T	$V_+ = \frac{V_o}{2}, \quad V_- = \frac{\Gamma_L V_o (t-2T)}{2T}$	$V_+ = \frac{V_o}{2}, \quad V_- = \Gamma_L V_+$
3T < t < 4T	$V_+ = \frac{V_o (4T-t)}{2T}, \quad V_- = \Gamma_L \frac{V_o}{2}$	$V_+ = \frac{V_o}{2}, \quad V_- = \Gamma_L V_+$
4T < t < 5T	$V_+ = 0, \quad V_- = \frac{\Gamma_L V_o}{2}$	$V_+ = \frac{V_o (5T-t)}{2T}, \quad V_- = \Gamma_L V_+$
5T < t < 6T	$V_+ = 0, \quad V_- = \frac{V_o \Gamma_L (6T-t)}{2T}$	$V_+ = 0, \quad V_- = 0$
t > 6T	$V_+ = 0, \quad V_- = 0$	$V_+ = 0, \quad V_- = 0$

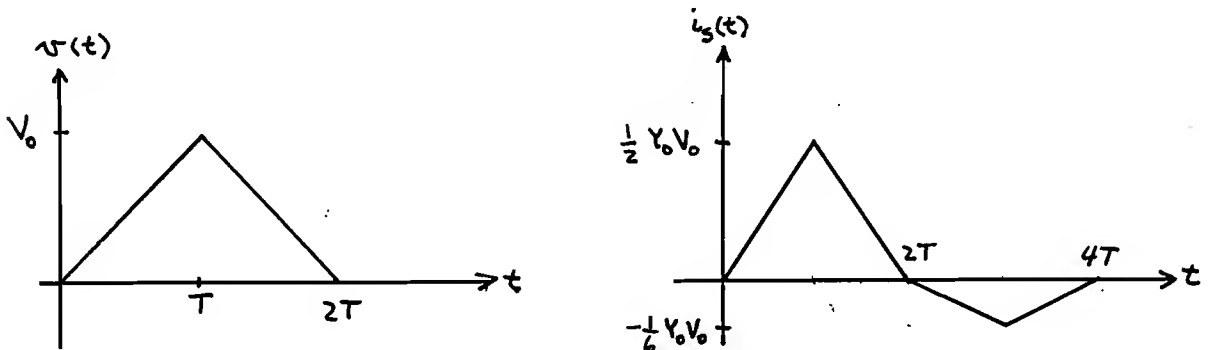


12. a)  $v(t) = V_0 u(t) \rightarrow i_s(t) = \frac{Y_0 V_0}{2} [u(t) - \Gamma_L u(t - 2T)]; \Gamma_L = \frac{1}{3}$





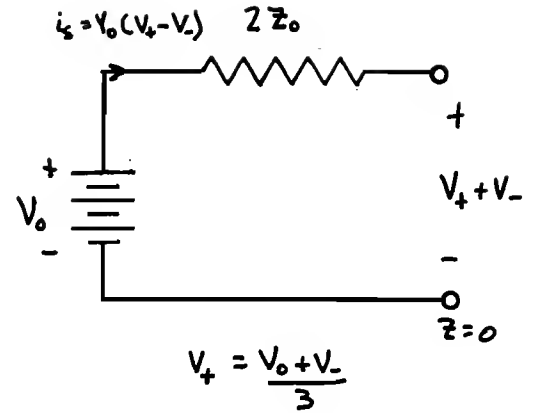
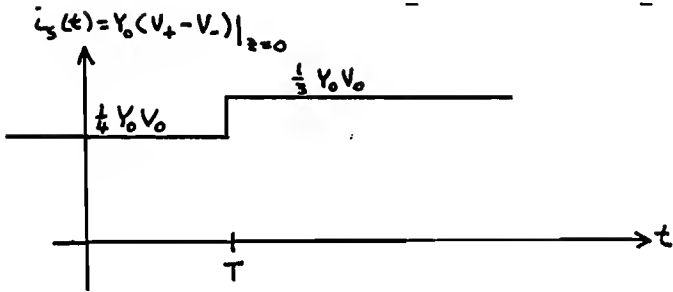
- c) Triangular voltage waveform is integral of  $v(t) = V_0 [u(t) - 2u(t-T) + u(t-2T)]$ . The source current is then the integral of  $i_s(t) = \frac{V_0}{2} [u(t) - 2u(t-T) + u(t-2T) - \Gamma_L [u(t-2T) - 2u(t-3T) + u(t-4T)]]$



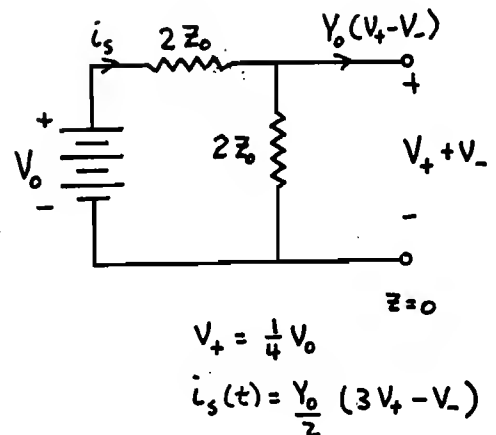
$$13. \quad v(t=0_-) = \frac{1}{2} V_o = V_+ + V_- \quad \rightarrow \quad V_+ = \frac{3}{8} V_o$$

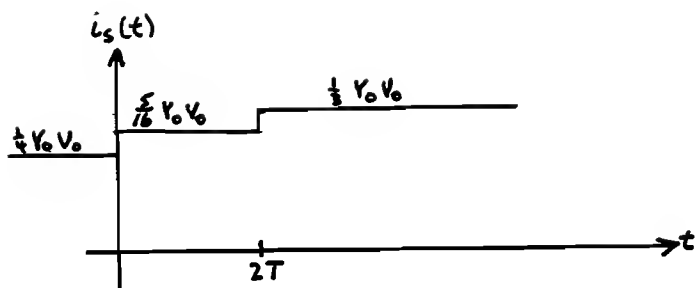
$$i(t=0_-) = \frac{V_o Y_o}{4} = Y_o (V_+ - V_-) \quad \rightarrow \quad V_- = \frac{1}{8} V_o$$

a) t	z = 0	z = \ell
0 < t < T	$V_+ = \frac{3}{8} V_o$ $V_- = \frac{1}{8} V_o$	$V_+ = \frac{3}{8} V_o$ $V_- = 0$
T < t < 2T	$V_+ = \frac{1}{3} V_o$ $V_- = 0$	$V_+ = \frac{3}{8} V_o$ $V_- = 0$



b) t	z = 0	z = \ell
0 < t < T	$V_+ = \frac{V_o}{4}$ $V_- = \frac{1}{8} V_o$	$V_+ = \frac{3}{8} V_o$ $V_- = \frac{1}{8} V_o$
T < t < 2T	$V_+ = \frac{V_o}{4}$ $V_- = \frac{1}{8} V_o$	$V_+ = \frac{V_o}{4}$ $V_- = \frac{V_o}{12}$
2T < t < 3T	$V_+ = \frac{V_o}{4}$ $V_- = \frac{V_o}{12}$	$V_+ = \frac{V_o}{4}$ $V_- = \frac{V_o}{12}$



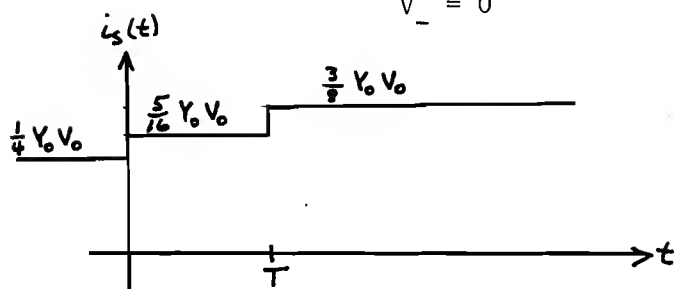


c)  $t$   $z = 0$   $z = \ell$

$0 < t < T$   $V_+ = \frac{V_0}{4}$   $V_+ = \frac{3}{8} V_0$

$T < t < 2T$   $V_+ = \frac{V_0}{4}$   $V_+ = \frac{V_0}{4}$

$V_- = 0$   $V_- = 0$



14. a) Inductive Load

Capacitive Load

$$v(t=0_-) = 0 = V_+ + V_-$$

$$v(t=0_-) = 0 = V_+ + V_-$$

$$i(t=0_-) = \frac{V_0}{R_s} = Y_0 (V_+ - V_-)$$

$$\rightarrow V_+ = -V_- = \frac{V_0 Z_0}{2R_s}$$

$$i(t=0_-) = \frac{V_0}{2Z_0} = Y_0 (V_+ - V_-)$$

$$\rightarrow V_+ = -V_- = \frac{V_0}{4}$$

b)  $t$   $z = 0$   $z = \ell$

$0 < t < T$   $V_+ = 0$   $V_+ = \frac{V_0 Z_0}{2R_s}$

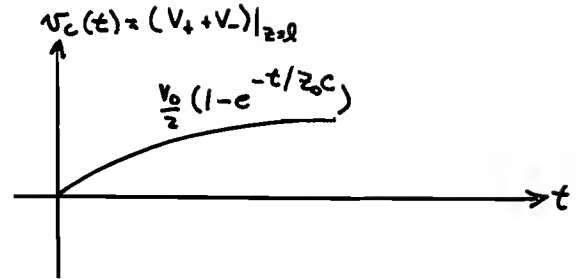
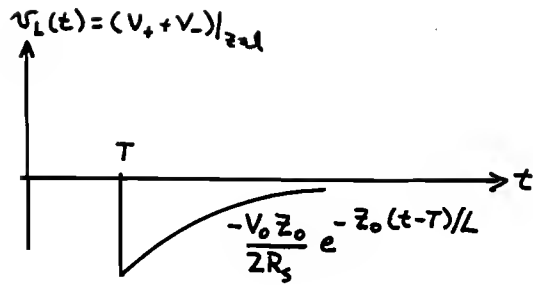
$V_- = \frac{-V_0 Z_0}{2R_s}$   $V_- = \frac{-V_0 Z_0}{2R_s}$

Inductive

$T < t < 2T$   $V_+ = 0$   $V_+ = 0$

$V_- = \frac{-V_0 Z_0}{2R_s}$   $V_- = \frac{-V_0 Z_0}{2R_s} e^{-Z_0(t-T)/L}$

# GUIDED ELECTROMAGNETIC WAVES



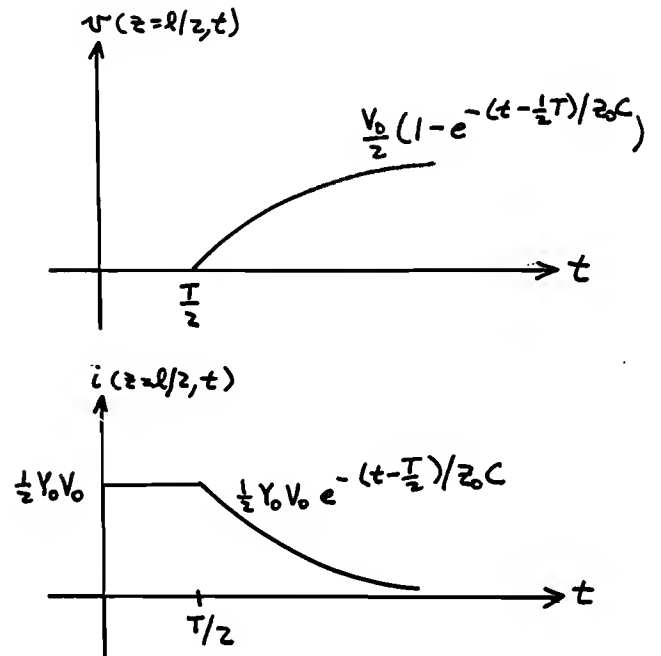
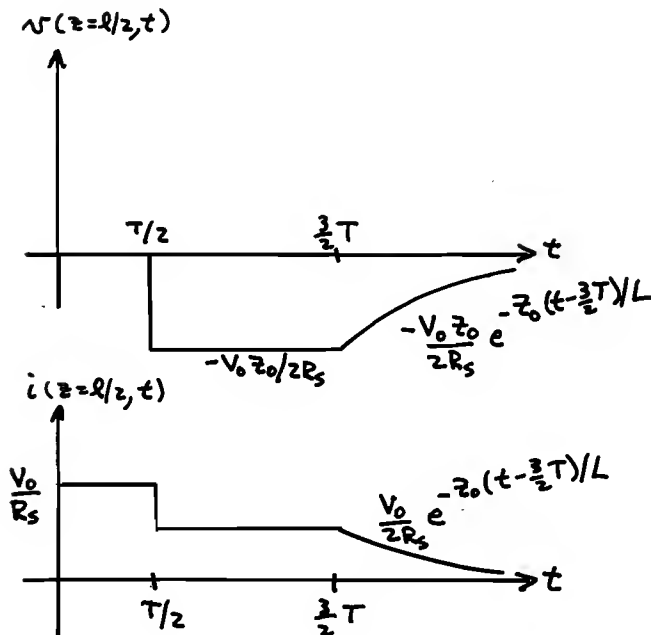
Capacitive

$t$	$z = 0$
$0 < t < T$	$V_+ = \frac{V_0}{4}$
	$V_- = \frac{-V_0}{4}$
$T < t < 2T$	$V_+ = \frac{V_0}{4}$
	$V_- = \frac{V_0}{2} \left( \frac{1}{2} - e^{-(t-T)/Z_0 C} \right)$

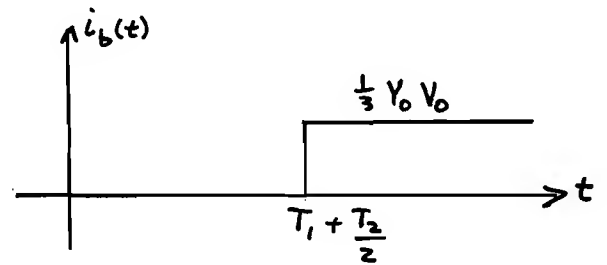
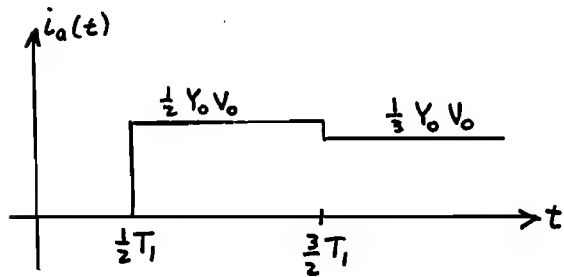
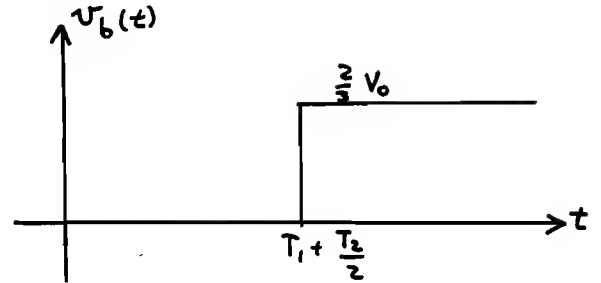
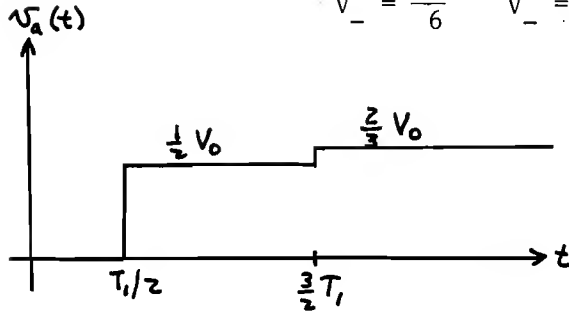
$z = l$
$V_+ = \frac{V_0}{4}$
$V_- = \frac{V_0}{2} \left( \frac{1}{2} - e^{-t/Z_0 C} \right)$
$V_+ = \frac{V_0}{4}$
$V_- = \frac{V_0}{2} \left( \frac{1}{2} - e^{-t/Z_0 C} \right)$

Inductive Load

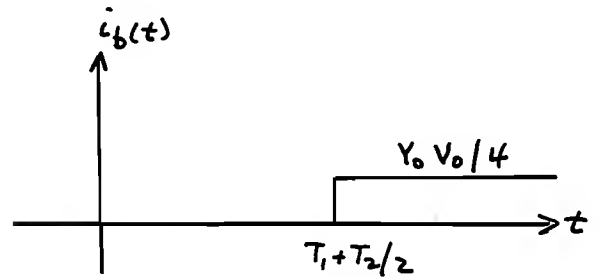
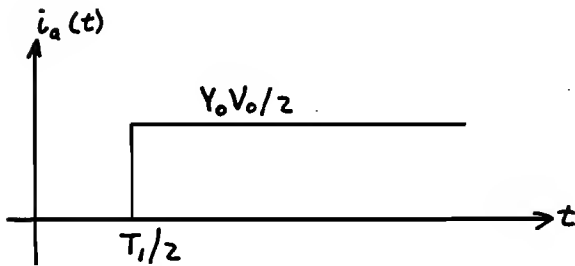
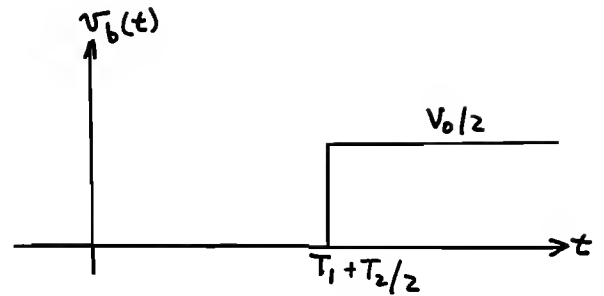
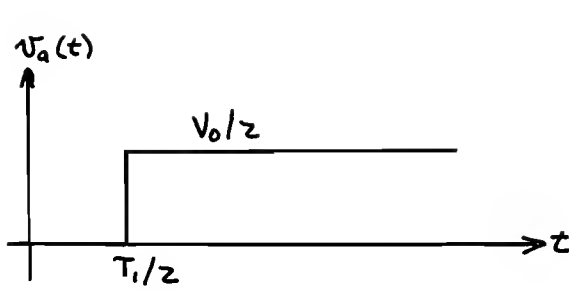
Capacitive Load



15. a) t	$z = 0$	$z = \ell_{1-}$	$z = \ell_{1+}$
$0 < t < T_1$	$V_+ = \frac{V_0}{2}$	$V_+ = 0$	$V_+ = 0$
	$V_- = 0$	$V_- = 0$	$V_- = 0$
$T_1 < t < 2T_1$	$V_+ = \frac{V_0}{2}$	$V_+ = \frac{V_0}{2}$	$V_+ = \frac{2}{3} V_0$
	$V_- = 0$	$V_- = \frac{V_0}{6}$	$V_- = 0$
$2T_1 < t < 3T_1$	$V_+ = \frac{V_0}{2}$	$V_+ = \frac{V_0}{2}$	$V_+ = \frac{2}{3} V_0$
	$V_- = \frac{V_0}{6}$	$V_- = \frac{V_0}{6}$	$V_- = 0$



b) t	$z = 0$	$z = \ell_{1-}$	$z = \ell_{1+}$
$0 < t < T_1$	$V_+ = \frac{V_0}{2}$	$V_+ = 0$	$V_+ = 0$
	$V_- = 0$	$V_- = 0$	$V_- = 0$
$T_1 < t < 2T_1$	$V_+ = \frac{V_0}{2}$	$V_+ = \frac{V_0}{2}$	$V_+ = \frac{V_0}{2}$
	$V_- = 0$	$V_- = 0$	$V_- = 0$



### Section 8.3

16. a)  $\hat{v}(z) = V_+ e^{-jkz} + V_- e^{jkz}$

$$\hat{i}(z) = Y_0 [V_+ e^{-jkz} - V_- e^{jkz}]$$

$$\hat{v}(z=0) = \hat{i}(z=0) jX \rightarrow V_+ + V_- = jXY_0 [V_+ - V_-]$$

$$\hat{v}(z=-\ell) = V_0 = V_+ e^{jk\ell} + V_- e^{-jk\ell}$$

$$V_+ = \frac{-V_0 (1 + jXY_0)}{[(1 - jXY_0)e^{-jk\ell} - (1 + jXY_0)e^{jk\ell}]}, \quad V_- = \frac{V_0 (1 - jXY_0)}{[(1 - jXY_0)e^{-jk\ell} - (1 + jXY_0)e^{jk\ell}]}$$

b) Resonance

$$(1 - jXY_0)e^{-jk\ell} = (1 + jXY_0)e^{jk\ell} \rightarrow \tan k\ell = -XY_0; \quad k = \frac{\omega}{c}$$

$$|X| = Z_0 \rightarrow \tan k\ell = \mp 1 \quad \begin{array}{l} \text{inductive} \\ \text{capacitive} \end{array} \rightarrow k\ell = \begin{array}{l} \frac{3}{4}\pi + n\pi \\ \frac{\pi}{4} + n\pi, \quad n = 0, 1, 2, \dots \end{array}$$



$$\omega_n = \begin{cases} \frac{c}{\ell} (n + \frac{3}{4})\pi & \text{inductive} \\ \frac{c}{\ell} (n + \frac{1}{4})\pi & \text{capacitive} \end{cases}$$

$$c) \quad \hat{i}(z = -\ell) = I_o = Y_o [V_+ e^{jk\ell} - V_- e^{-jk\ell}]$$

$$V_+ = \frac{I_o Z_o (1 + jXY_o)}{[(1 - jXY_o)e^{-jk\ell} + (1 + jXY_o)e^{jk\ell}]}; \quad V_- = \frac{-I_o Z_o (1 - jXY_o)}{[(1 - jXY_o)e^{-jk\ell} + (1 + jXY_o)e^{jk\ell}]}$$

Resonance

$$(1 - jXY_o)e^{-jk\ell} + (1 + jXY_o)e^{jk\ell} = 0 \rightarrow \tan k\ell = \frac{1}{XY_o}$$

$$|X| = Z_o \rightarrow \tan k\ell = \pm 1 \quad \begin{matrix} \text{inductive} \\ \text{capacitive} \end{matrix} \rightarrow \omega_n = \begin{cases} \frac{c}{\ell} (n + \frac{1}{4})\pi \\ \frac{c}{\ell} (n + \frac{3}{4})\pi \end{cases} \quad n = 0, 1, 2, \dots$$

$$17. \quad a) \quad R = \frac{1}{\sigma_w \delta 2\pi a} + \frac{1}{\sigma_w \delta 2\pi b}, \quad G = \frac{2\pi\sigma}{\ln \frac{b}{a}}$$

$$b) \quad \alpha = \frac{1}{2} [RY_o + GZ_o]; \quad Y_o = \frac{1}{Z_o} = \sqrt{\frac{\epsilon}{\mu}} \frac{2\pi}{\ln \frac{b}{a}}$$

$$= \frac{1}{2} \left\{ \frac{\sqrt{\epsilon/\mu} (\frac{1}{a} + \frac{1}{b})}{\sigma_w \delta \ln \frac{b}{a}} + \sigma \sqrt{\frac{\mu}{\epsilon}} \right\}$$

$$c) \quad \sigma = 0 \rightarrow \alpha = \frac{\frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} (\frac{1}{a} + \frac{1}{b})}{\sigma_w \delta \ln \frac{b}{a}}$$

$$\frac{d\alpha}{da} = \frac{\frac{1}{2} \sqrt{\frac{\epsilon}{\mu}}}{\sigma_w \delta} \left[ \frac{-1}{a^2 \ln \frac{b}{a}} + \frac{(\frac{1}{a} + \frac{1}{b})}{(\ln \frac{b}{a})^2} \frac{a}{b} \frac{b}{a^2} \right] = 0$$

$$\frac{a(\frac{1}{a} + \frac{1}{b})}{\ln \frac{b}{a}} = 1 \rightarrow \ln \frac{b}{a} = 1 + \frac{a}{b} \rightarrow \frac{b}{a} \approx 3.6$$

$$18. \quad a) \quad \hat{v}(z) = V_+ e^{-jkz} + V_- e^{jkz}$$

$$\hat{i}(z) = Y_o [V_+ e^{-jkz} - V_- e^{jkz}]$$

$$\hat{v}(z = -\ell) = V_o = V_+ e^{jk\ell} + V_- e^{-jk\ell}$$

$$\hat{v}(z=0) = \hat{i}(z=0) R_L \rightarrow V_+ + V_- = Y_o R_L (V_+ - V_-)$$

$$V_+ = \frac{-V_o (1 + Y_o R_L)}{[e^{-jk\ell} (1 - Y_o R_L) - (1 + Y_o R_L) e^{jk\ell}]}; V_- = \frac{V_o (1 - Y_o R_L)}{[e^{-jk\ell} (1 - Y_o R_L) - (1 + Y_o R_L) e^{jk\ell}]}$$

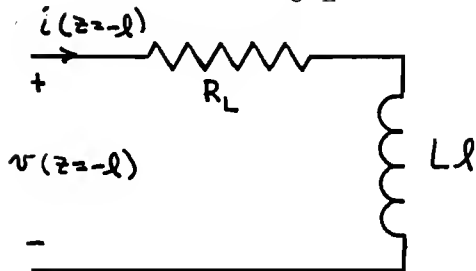
$$b) \lim_{k\ell \ll 1} V_+ \approx \frac{V_o (1 + Y_o R_L)}{2Y_o R_L + 2jk\ell}, V_- \approx \frac{-V_o (1 - Y_o R_L)}{2Y_o R_L + 2jk\ell}$$

$$\begin{aligned} \hat{v}(z) &\approx \frac{V_o}{2(Y_o R_L + jk\ell)} [(1 + Y_o R_L)(1 - jkz) - (1 - Y_o R_L)(1 + jkz)] \\ &\approx \frac{V_o}{(Y_o R_L + jk\ell)} [Y_o R_L - jkz] \end{aligned}$$

$$\begin{aligned} \hat{i}(z) &\approx \frac{Y_o V_o}{2(Y_o R_L + jk\ell)} [(1 + Y_o R_L)(1 - jkz) + (1 - Y_o R_L)(1 + jkz)] \\ &\approx \frac{Y_o V_o}{(Y_o R_L + jk\ell)} [1 - jkz Y_o R_L] \end{aligned}$$

$$c) Y_o R_L \ll k\ell$$

$$\hat{i}(z = -\ell) = \frac{Y_o V_o (1 + jk\ell Y_o R_L)}{Y_o R_L + jk\ell} \rightarrow V_o = \frac{\hat{i}(z = -\ell) [Y_o R_L + jk\ell] Z_o}{1 + jk\ell Y_o R_L}$$



$$\approx \hat{i}(z = -\ell) Z_o [Y_o R_L + jk\ell] [1 - jk\ell Y_o R_L]$$

$$\approx \hat{i}(z = -\ell) Z_o [Y_o R_L + jk\ell]$$

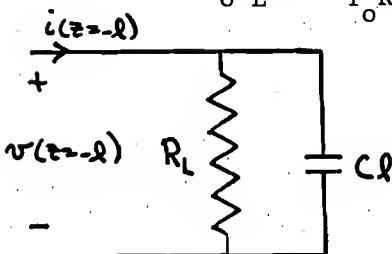
$$\approx \hat{i}(z = -\ell) [R_L + j\sqrt{L/C} \omega \sqrt{LC} \ell]$$

$$\approx \hat{i}(z = -\ell) [R_L + j\omega L\ell]$$

$$Y_o R_L \gg k\ell$$

$$\hat{i}(z=-\ell) \approx \frac{Y_o V_o (1 + jk\ell Y_o R_L)}{Y_o R_L (1 + \frac{jk\ell}{Y_o R_L})} \approx \frac{V_o}{R_L} (1 + jk\ell Y_o R_L) (1 - \frac{jk\ell}{Y_o R_L})$$

$$\approx V_o (\frac{1}{R_L} + jk\ell Y_o)$$

$$\approx V_o (\frac{1}{R_L} + j\omega C\ell)$$


## Section 8.4

19. a)  $Z_n(z=0) = 2(1-j)$ ,  $Z_n(z=-\frac{\lambda}{4}) = \frac{1}{2(1-j)} \rightarrow Y_n(z=-\frac{\lambda}{4}) = 2(1-j)$

$$Y_T(z=-\frac{\lambda}{4}) = jB + \frac{2(1-j)}{50} \rightarrow B = .04, R_s = 25$$

b)  $\langle P_L \rangle = \frac{1}{4} \frac{V_o^2}{R_s}$

20. a) Switch open  $\rightarrow Z(z=-\frac{\lambda}{2}) = 400 \rightarrow \langle P \rangle = \frac{V_o^2}{1600}$

Switch closed  $\rightarrow Z(z=-\frac{\lambda}{2}) = 800 \rightarrow \langle P \rangle = \frac{V_o^2}{2400}$

b) Switch open  $\rightarrow \langle P_L \rangle = \frac{1}{2} \langle P \rangle = \frac{V_o^2}{3200}$

Switch closed  $\rightarrow \langle P_L \rangle = \frac{V_o^2}{7200}$

c) Switch open  $\rightarrow Z_o = 50$  line,  $VSWR = \frac{R_L}{Z_o} = 2$

$Z_o = 100$  line,  $VSWR = \frac{Z_o}{25} = 4$

Switch closed  $\rightarrow Z_o = 50$  line,  $VSWR = \frac{R_L}{Z_o} = 2$

$Z_o = 100$  line,  $VSWR = \frac{Z_o}{12.5} = 8$

21. a) and b)  $i(t) = \text{Re} \{ \hat{I} e^{j\omega t} \}$ ;  $\hat{I} = |\hat{I}| e^{j\phi} \rightarrow i(t) = |\hat{I}| \cos(\omega t + \phi)$

$$\hat{I} = \frac{V_o}{Z_o Z_n(-\ell)}$$

$\ell$	$Z_n(-\ell)$	$\hat{\mathbf{I}}$	$ \hat{\mathbf{I}} $	$\phi$	$\langle P \rangle = \frac{1}{2} V_o  \hat{\mathbf{I}}  \cos \phi$
$\frac{\lambda}{8}$	$.2 - .4j$	$\frac{V_o}{50(.2 - .4j)}$	$\frac{V_o}{50\sqrt{.2}}$	$63.4^\circ$	$\frac{V_o^2}{100\sqrt{.2}} (.447)$
$\frac{\lambda}{4}$	$.2 + .4j$	$\frac{V_o}{50(.2 + .4j)}$	$\frac{V_o}{50\sqrt{.2}}$	$-63.4^\circ$	$\frac{V_o^2}{100\sqrt{.2}} (.447)$
$\frac{3\lambda}{8}$	$1 + 2j$	$\frac{V_o}{50(1+2j)}$	$\frac{V_o}{50\sqrt{5}}$	$-63.4^\circ$	$\frac{V_o^2}{100\sqrt{5}} (.447)$
$\frac{\lambda}{2}$	$1 - 2j$	$\frac{V_o}{50(1-2j)}$	$\frac{V_o}{50\sqrt{5}}$	$63.4^\circ$	$\frac{V_o^2}{100\sqrt{5}} (.447)$

$$c) \quad \Gamma_L = \frac{Z_{nL} - 1}{Z_{nL} + 1} = \frac{-2j}{2(1 - j)} = \frac{-j}{1 - j}$$

$$|\Gamma_L| = \frac{1}{\sqrt{2}} \rightarrow \text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \approx 5.83$$

$$22. \quad a) \quad \hat{v}(z) = V_+ e^{-jkz} + V_- e^{jkz}$$

$$\hat{i}(z) = Y_o [V_+ e^{-jkz} - V_- e^{jkz}]$$

$$\hat{v}(z=0) = \hat{i}(z=0) R_L \rightarrow V_+ + V_- = R_L Y_o (V_+ - V_-) \rightarrow \frac{V_-}{V_+} = \Gamma_L = \frac{R_L Y_o - 1}{R_L Y_o + 1} = \frac{1}{3}$$

$$\hat{v}(z=-\ell) = V_o = V_+ e^{jk\ell} + V_- e^{-jk\ell}; \quad k\ell = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \frac{\pi}{2}$$

$$= j(V_+ - V_-)$$

$$= \frac{2}{3} jV_+$$

$$V_+ = -\frac{3}{2} jV_o, \quad V_- = -\frac{1}{2} jV_o$$

$$\hat{v}(z) = -\frac{jV_o}{2} [3e^{-jkz} + e^{jkz}]$$

$$\hat{i}(z) = -j \frac{V_o Y_o}{2} [3e^{-jkz} - e^{jkz}]$$

$$b) \quad VSWR = \frac{R_L}{Z_o} = 2$$

$$\begin{aligned} c) \quad |\hat{v}(z)|^2 &= \left(\frac{V_o}{2}\right)^2 [3e^{-jkz} + e^{jkz}][3e^{jkz} + e^{-jkz}] \\ &= \left(\frac{V_o}{2}\right)^2 [10 + 3(e^{-2jkz} + e^{2jkz})] \\ &= \left(\frac{V_o}{2}\right)^2 [10 + 6\cos 2kz] \end{aligned}$$

$$|\hat{v}(z)|_{\max} \rightarrow kz = -n\pi, \quad n = 0, 1, 2, \dots$$

$$\text{Since } -\frac{\lambda}{4} < z < 0 \rightarrow n = 0 \rightarrow z = 0 \rightarrow |\hat{v}(z)|_{\max} = 2V_o$$

$$|\hat{v}(z)|_{\min} \rightarrow kz = -\frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \dots$$

$$n = 0 \rightarrow z = -\frac{\lambda}{4} \rightarrow |\hat{v}(z)|_{\min} = V_o$$

$$\text{Check: } VSWR = \frac{|\hat{v}(z)|_{\max}}{|\hat{v}(z)|_{\min}} = 2$$

$$23. \quad VSWR = 3, \quad Z_o = 100, \quad \frac{\lambda}{2} = .5 \text{ meters}, \quad d_{\min} = .2 \text{ meters}$$

$$kd_{\min} = \frac{2\pi d_{\min}}{\lambda} = .4\pi$$

$$|\Gamma_L| = \frac{VSWR - 1}{VSWR + 1} = \frac{1}{2}, \quad \phi = \pi \left[ \frac{4d_{\min}}{\lambda} - 1 \right] = -.2\pi$$

$$\Gamma_L = |\Gamma_L| e^{j\phi} = \frac{1}{2} e^{-j(.2\pi)}$$

$$Z_L = Z_o \frac{[VSWR - j \tan \frac{\phi}{2}]}{[1 - j VSWR \tan \frac{\phi}{2}]}; \quad \tan \frac{\phi}{2} = -.3249$$

$$= \frac{100[3 + .3249j]}{[1 + j(3)(.3249)]}$$

$$= 170.08 - 133.29j$$

$$24. \quad a) \quad A \rightarrow Z_{nL} = 2(1 - j), \quad B \rightarrow Y_{nL} = .25 + .25j$$

$$C \rightarrow Y_n(-\ell_1) = 1 + 1.6j, \quad \ell_1 = \ell_D = .137\lambda$$

$$Y_{n2} = -1.6j \rightarrow \ell_2 = .089\lambda$$

$$D \rightarrow Y_n(-\ell_1) = 1 - 1.6j, \quad \ell_1 = \ell_D + \ell_C = .279\lambda$$

$$Y_{n2} = +1.6j \rightarrow \ell_2 = .411\lambda$$

$$\ell_1 = .137\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .089\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .279\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .411\lambda + \frac{m\lambda}{2}$$

$$b) \quad \ell_1 = .25\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .074\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .3755\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .426\lambda + \frac{m\lambda}{2}$$

$$c) \quad \ell_1 = .039\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .177\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .25\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .324\lambda + \frac{m\lambda}{2}$$

$$d) \quad \ell_1 = .294\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .089\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .446\lambda + \frac{n\lambda}{2}, \quad \ell_2 = .411\lambda + \frac{m\lambda}{2}$$

$$25. \quad a) \quad Z_{nL} = 2(1 - j) \rightarrow Y_{nL} = .25 + .25j$$

$$Y_{a1} = .25 - j.3375$$

$$Y_{a2} = .25 - j1.65$$

$$Y_1 = j\text{Im}(Y_a - Y_{nL}) = \begin{cases} -j.5875 \rightarrow \ell_1 = .166\lambda \\ -j1.90 \rightarrow \ell_1 = .077\lambda \end{cases}$$

$$Y_{b1} = 1 - j1.6$$

$$Y_{b2} = 1 + j3.6$$

$$Y_2 = -j\text{Im}Y_b = \begin{cases} j1.6 \rightarrow \ell_2 = .411\lambda \\ -j3.6 \rightarrow \ell_2 = .043\lambda \end{cases}$$

$$\ell_1 = .166\lambda + \frac{n\lambda}{2}, \ell_2 = .411\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .077\lambda + \frac{n\lambda}{2}, \ell_2 = .043\lambda + \frac{m\lambda}{2}$$

$$b) \quad Z_{nL} = 1 + 2j \rightarrow Y_{nL} = .2 - j.4 = Y_{a1}$$

$$Y_{a2} = .2 - j1.6$$

$$Y_1 = j\text{Im}(Y_a - Y_{nL}) = \begin{cases} j0 \rightarrow \ell_1 = .25\lambda \\ -j1.2 \rightarrow \ell_1 = .11\lambda \end{cases}$$

$$Y_{b1} = 1 - j2.0, Y_{b2} = 1 + j3.8$$

$$Y_2 = -j\text{Im}Y_b = \begin{cases} j2.0 \rightarrow \ell_2 = .426\lambda \\ -j3.8 \rightarrow \ell_2 = .041\lambda \end{cases}$$

$$\ell_1 = .25\lambda + \frac{n\lambda}{2}, \ell_2 = .426\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .11\lambda + \frac{n\lambda}{2}, \ell_2 = .041\lambda + \frac{m\lambda}{2}$$

$$c) \quad Z_{nL} = 1 - j.5 \rightarrow Y_{nL} = .8 + j.4$$

$$Y_{a1} = .8 - j.025$$

$$Y_{a2} = .8 - j2.0$$

$$Y_1 = j\text{Im}(Y_a - Y_{nL}) = \begin{cases} -j.425 \rightarrow \ell_1 = .187\lambda \\ -j2.4 \rightarrow \ell_1 = .063\lambda \end{cases}$$

$$Y_{b1} = 1 - j.25, Y_{b2} = 1 + j2.2$$

$$Y_2 = -j\text{Im}Y_b = \begin{cases} j.25 \rightarrow \ell_2 = .289\lambda \\ -j2.2 \rightarrow \ell_2 = .068\lambda \end{cases}$$

$$\ell_1 = .187\lambda + \frac{n\lambda}{2}, \ell_2 = .289\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .063\lambda + \frac{n\lambda}{2}, \ell_2 = .068\lambda + \frac{m\lambda}{2}$$

$$d) \quad Z_{nL} = .5 + j1 \rightarrow Y_{nL} = .4 - j.8$$

$$Y_{a1} = .4 - j.2, Y_{a2} = .4 - j1.8$$

$$Y_1 = j\text{Im}(Y_a - Y_{nL}) = \begin{cases} +j.6 \rightarrow \ell_1 = .334\lambda \\ -j1.0 \rightarrow \ell_1 = .125\lambda \end{cases}$$

$$Y_{b1} = 1 - j, Y_{b2} = 1 + j3.0$$

$$Y_2 = -j\text{Im}Y_b = \begin{cases} j \rightarrow \ell_2 = .375\lambda \\ -j3.0 \rightarrow \ell_2 = .051\lambda \end{cases}$$

$$\ell_1 = .334\lambda + \frac{n\lambda}{2}, \ell_2 = .375\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .125\lambda + \frac{n\lambda}{2}, \ell_2 = .051\lambda + \frac{m\lambda}{2}$$

26. a) Inductive

$$Z_n(z=0) = 1 + j$$

$$Z_n(z = -\frac{\lambda}{4}) = \frac{1}{1+j} = \frac{(1-j)}{2}$$

$$Z(z = -\frac{\lambda}{4}) = \frac{Z_o}{2} (1-j)$$

$$b) \hat{i}(z = -\frac{\lambda}{4}) = \frac{2V_o}{Z_o(1-j)} = \frac{V_o(1+j)}{Z_o}$$

$$i(z = -\frac{\lambda}{4}, t) = \text{Re} \hat{i}(z = -\frac{\lambda}{4}) e^{j\omega t} \\ = \frac{\sqrt{2} V_o}{Z_o} \cos(\omega t + 45^\circ)$$

c)  $\omega_o \rightarrow 2\omega_o$

$$Z_n(z=0) = 1 + 2j$$

$$Z_n(z = -\frac{\lambda}{4}) = \frac{1}{1+2j} = \frac{(1-2j)}{5}$$

$$Z(z = -\frac{\lambda}{4}) = \frac{Z_o(1-2j)}{5}$$

Capacitive

$$Z_n(z=0) = 1 - j$$

$$Z_n(z = -\frac{\lambda}{4}) = \frac{1}{1-j} = \frac{1+j}{2}$$

$$Z(z = -\frac{\lambda}{4}) = \frac{Z_o}{2} (1+j)$$

$$\hat{i}(z = -\frac{\lambda}{4}) = \frac{2V_o}{Z_o(1+j)} = \frac{V_o(1-j)}{Z_o}$$

$$i(z = -\frac{\lambda}{4}, t) = \text{Re} \hat{i}(z = -\frac{\lambda}{4}) e^{j\omega t} \\ = \frac{\sqrt{2} V_o}{Z_o} \cos(\omega t - 45^\circ)$$

$$Z_n(z=0) = 1 - \frac{1}{2} j$$

$$Z_n(z = -\frac{\lambda}{4}) = \frac{1}{1 - \frac{1}{2} j} = \frac{1 + \frac{1}{2} j}{1.25}$$

$$Z(z = -\frac{\lambda}{4}) = \frac{Z_o(1 + \frac{1}{2} j)}{1.25}$$



Inductive

$$\hat{z}(z = -\frac{\lambda}{4}) = \frac{5V_o}{Z_o(1-2j)} = \frac{V_o(1+2j)}{Z_o}$$

$$i(z = -\frac{\lambda}{4}, t) = \frac{V_o \sqrt{5}}{Z_o} \cos(\omega t + 63.43^\circ)$$

$$d) \quad \omega_o \rightarrow \frac{1}{2} \omega_o$$

$$Z_n(z=0) = 1 + \frac{j}{2}$$

$$Z_n(z = -\frac{\lambda}{4}) = \frac{1}{1 + \frac{j}{2}} = \frac{(1 - \frac{j}{2})}{1.25}$$

$$Z(z = -\frac{\lambda}{4}) = \frac{Z_o(1 - \frac{j}{2})}{1.25}$$

$$\hat{z}(z = -\frac{\lambda}{4}) = \frac{V_o}{Z_o} (1 + \frac{j}{2})$$

$$i(z = -\frac{\lambda}{4}, t) = \frac{V_o \sqrt{1.25}}{Z_o} \cos(\omega t + 26.57^\circ)$$

Capacitive

$$\hat{z}(z = -\frac{\lambda}{4}) = \frac{(1 - \frac{j}{2})V_o}{Z_o}$$

$$i(z = -\frac{\lambda}{4}, t) = \frac{V_o \sqrt{1.25}}{Z_o} \cos(\omega t - 26.57^\circ)$$

$$Z_n(z=0) = 1 - 2j$$

$$Z_n(z = -\frac{\lambda}{4}) = \frac{1}{1-2j} = \frac{(1+2j)}{5}$$

$$Z(z = -\frac{\lambda}{4}) = \frac{Z_o(1+2j)}{5}$$

$$\hat{z}(z = -\frac{\lambda}{4}) = \frac{V_o(1-2j)}{Z_o}$$

$$i(z = -\frac{\lambda}{4}, t) = \frac{V_o \sqrt{5}}{Z_o} \cos(\omega t - 63.43^\circ)$$

### Section 8.6

$$27. \quad a) \quad \nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}_f + \epsilon \frac{\partial \bar{E}}{\partial t}; \quad \frac{\partial \bar{J}_f}{\partial t} = \omega_p^2 \epsilon \bar{E}$$

$$\epsilon \nabla \cdot \bar{E} = 0$$

$$\mu \nabla \cdot \bar{H} = 0$$

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \bar{H}) = -\mu \left[ \omega_p^2 \epsilon \bar{E} + \epsilon \frac{\partial^2 \bar{E}}{\partial t^2} \right]$$

$$\nabla^2 \bar{E} = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} + \frac{\omega_p^2}{c^2} \bar{E}$$

$$\nabla \times (\nabla \times \bar{H}) = \nabla(\nabla \cdot \bar{H}) - \nabla^2 \bar{H} = \nabla \times \bar{J}_f + \epsilon \frac{\partial}{\partial t} \nabla \times \bar{E}$$

$$= \nabla \times \bar{J}_f - \epsilon \mu \frac{\partial^2 \bar{H}}{\partial t^2}$$

$$\begin{aligned}\nabla^2 \frac{\partial \bar{H}}{\partial t} &= -\nabla \times \frac{\partial \bar{J}_f}{\partial t} + \epsilon \mu \frac{\partial^3 \bar{H}}{\partial t^3} = -\omega_p^2 \epsilon \nabla \times \bar{E} + \epsilon \mu \frac{\partial^3 \bar{H}}{\partial t^3} \\ &= \frac{\omega_p^2}{c^2} \frac{\partial \bar{H}}{\partial t} + \frac{1}{c^2} \frac{\partial^3 \bar{H}}{\partial t^3}\end{aligned}$$

$$\nabla^2 \bar{H} = \frac{1}{c^2} \frac{\partial^2 \bar{H}}{\partial t^2} + \frac{\omega_p^2}{c^2} \bar{H}$$

Solutions the same as in text with  $k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2 - \omega_p^2}{c^2}$

TM - Eqs. (13), (18), (19), (22)

TE - Eqs. (30), (32)

$$\text{b) } k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b} \rightarrow k_z = \left[ \frac{\omega^2 - \omega_p^2}{c^2} - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \right]^{1/2}$$

$$\omega_{co} = \left[ \omega_p^2 + \left( \frac{\pi c}{a} \right)^2 \right]^{1/2}$$

$$\text{c) } v_p = \frac{\omega}{k_z}, v_g = \frac{d\omega}{dk_z} = \frac{k_z c^2}{\omega}$$

d) TM [From (42) and (43)]

$$\langle P_z \rangle = \frac{\omega \epsilon k_z a b E_o^2}{8(k_x^2 + k_y^2)} \quad \omega > \omega_{co}$$

TE [From (47)]

$$\langle P_z \rangle = \begin{cases} \frac{\omega \mu k_z a b H_o^2}{8(k_x^2 + k_y^2)} & m, n \neq 0 \\ \frac{\omega \mu k_z a b H_o^2}{4(k_x^2 + k_y^2)} & m \text{ or } n = 0 \end{cases} \quad \omega > \omega_{co}$$

e) From (54)

$$\alpha = \frac{2 \left( \frac{\pi}{a} \right)^2 \left[ b + \frac{a}{2} \left( \frac{\omega_a^2}{\omega_c^2} \right) \right]}{\omega \mu a b k_z \sigma_w \delta}; \quad k_z = \left[ \frac{\omega^2 - \omega_p^2}{c^2} - \left( \frac{\pi}{a} \right)^2 \right]^{1/2}$$

28. a)

$$\langle P_z \rangle = \frac{\omega \epsilon k_z a b E_o^2}{8(k_x^2 + k_y^2)}; \quad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}$$

$$|\hat{K}_z(x, y=0)| = |\hat{K}_z(x, y=b)| = \frac{k_y k_x^2 E_o \sin k_x x}{j\omega\mu(k_x^2 + k_y^2)}$$

$$|\hat{K}_z(x=0, y)| = |\hat{K}_z(x=a, y)| = \frac{k_x k_y^2 E_o \sin k_y y}{j\omega\mu(k_x^2 + k_y^2)}$$

$$\langle P_d(x=0, y) \rangle = \langle P_d(x=a, y) \rangle = \frac{1}{2} \operatorname{Re}(\hat{E} \cdot \hat{K}^*) = \frac{1}{2} \frac{|\hat{K}_z(x=0, y)|^2}{\sigma_w \delta}$$

$$\langle P_d(x, y=0) \rangle = \langle P_d(x, y=b) \rangle = \frac{1}{2} \operatorname{Re}(\hat{E} \cdot \hat{K}^*) = \frac{1}{2} \frac{|\hat{K}_z(x, y=0)|^2}{\sigma_w \delta}$$

$$\langle P_{dL} \rangle = \int_0^b 2\langle P_d(x=0, y) \rangle dy + \int_0^a 2\langle P_d(x, y=0) \rangle dx$$

$$= \frac{k_o^4 E_o^2}{\omega^2 \mu^2 (k_x^2 + k_y^2)^2 \sigma_w \delta} \left\{ \int_0^b k_x^2 \sin^2 k_y y dy + \int_0^a k_y^2 \sin^2 k_x x dx \right\}$$

$$= \frac{\omega^2 \epsilon^2 E_o^2}{\sigma_w \delta (k_x^2 + k_y^2)^2} \left\{ \frac{k_x^2 b}{2} + \frac{k_y^2 a}{2} \right\}$$

$$b) \quad \alpha = \frac{1}{2} \frac{\langle P_{dL} \rangle}{\langle P \rangle} = \frac{2\omega\epsilon(bk_x^2 + ak_y^2)}{\sigma_w \delta k_z a b (k_x^2 + k_y^2)}$$

 29. a) TE
Electric Field Lines

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{k_x \tan k_x x}{k_y \tan k_y y} \rightarrow \tan k_y y \, d(k_y y) = -\tan k_x x \, d(k_x x)$$

$$-\ln \cos k_y y = \ln \cos k_x x + \text{constant} \rightarrow \cos k_x x \cos k_y y = \text{constant}$$

Magnetic Field Lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{k_y}{k_x} \frac{\tan k_y y}{\tan k_x x} \rightarrow \frac{\cot k_y y \, d(k_y y)}{k_y^2} = \frac{\cot k_x x \, d(k_x x)}{k_x^2}$$

$$\frac{1}{k_y^2} \ln \sin k_y y = \frac{1}{k_x^2} \ln \sin k_x x + \text{constant}$$

$$\frac{(\sin k_x x)^{(k_y/k_x)^2}}{\sin k_y y} = \text{constant}$$

TM

Electric Field Lines

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{k_y}{k_x} \frac{\tan k_x x}{\tan k_y y} \rightarrow \frac{\tan k_y y \, d(k_y y)}{k_y^2} = \frac{\tan k_x x \, d(k_x x)}{k_x^2}$$

$$\frac{1}{k_y^2} \ln \cos k_y y = \frac{1}{k_x^2} \ln \cos k_x x + \text{constant}$$

$$\frac{(\cos k_x x)^{(k_y/k_x)^2}}{\cos k_y y} = \text{constant}$$

Magnetic Field Lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -\frac{k_x}{k_y} \frac{\cot k_x x}{\cot k_y y} \rightarrow \cot k_y y \, d(k_y y) = -\cot k_x x \, d(k_x x)$$

$$\ln \sin k_y y = -\ln \sin k_x x + \text{constant}$$

$$\sin k_x x \sin k_y y = \text{constant}$$

$$b) \quad \hat{K}(x, y=0) = H_0 \left[ \bar{i}_x \cos k_x x - \bar{i}_z \frac{j k_z k_x \sin k_x x}{k_x^2 + k_y^2} \right]$$

$$\hat{K}(x, y=b) = -H_0 \cos n\pi \left[ \bar{i}_x \cos k_x x - \bar{i}_z \frac{j k_z k_x}{k_x^2 + k_y^2} \sin k_x x \right]$$

$$\hat{K}(x=0, y) = H_0 \left[ -\bar{i}_y \cos k_y y + \bar{i}_z \frac{j k_z k_y}{k_x^2 + k_y^2} \sin k_y y \right]$$

$$\hat{K}(x=a, y) = -H_0 \cos m\pi \left[ -\bar{i}_y \cos k_y y + \bar{i}_z \frac{j k_y k_z}{k_x^2 + k_y^2} \sin k_y y \right]$$

On  $y = 0$  and  $y = b$  surfaces

$$\frac{dz}{dx} = \frac{K_z}{K_x} = \frac{-k_z k_x}{k_x^2 + k_y^2} \tan k_x x \tan k_z z$$

$$\cot k_z z \, d(k_z z) = \frac{-k_z^2}{k_x^2 + k_y^2} \tan k_x x \, d(k_x x)$$

$$\ln \sin k_z z = \frac{+k_z^2}{k_x^2 + k_y^2} \ln \cos k_x x + \text{constant}$$

$$\frac{(\cos k_x x)^{k_z^2/(k_x^2 + k_y^2)}}{\sin k_z z} = \text{constant}$$

On  $x = 0$  and  $x = a$  surfaces

$$\frac{dz}{dy} = \frac{-k_z k_y}{k_x^2 + k_y^2} \tan k_y y \tan k_z z$$

$$\frac{(\cos k_y y)^{k_z^2/(k_x^2 + k_y^2)}}{\sin k_z z} = \text{constant}$$

c) TM - At  $x = 0$

$$\hat{\sigma}_f(x=0, y) = \frac{-j k_z k_x}{k_x^2 + k_y^2} \epsilon E_0 \sin k_y y$$

$$\hat{K}_z(x=0, y) = \frac{k_x k_z^2 E_0}{j \omega \mu (k_x^2 + k_y^2)} \sin k_y y$$

$$\nabla_{\Sigma} \cdot \bar{K} = \frac{\partial K_z}{\partial z} \rightarrow -j k_z \hat{K}_z$$

$$\frac{\partial \sigma_f}{\partial t} \rightarrow j \omega \hat{\sigma}_f$$

$$\nabla_{\Sigma} \cdot \bar{K} + \frac{\partial \sigma_f}{\partial t} = \frac{-k_z k_x}{(k_x^2 + k_y^2)} E_0 \sin k_y y \left[ \frac{k^2}{\omega \mu} - \omega \epsilon \right] = 0$$

TE - At  $x = 0$

$$\hat{\sigma}_f(x=0, y) = \frac{-k_y k_z^2 H_0}{j\omega(k_x^2 + k_y^2)} \sin k_y y$$

$$\hat{K}(x=0, y) = H_0 \left[ \bar{i}_z \frac{j k_z k_y}{k_x^2 + k_y^2} \sin k_y y - \bar{i}_y \cos k_y y \right]$$

$$\nabla_{\Sigma} \cdot \bar{K} \rightarrow -j k_z \hat{K}_z + \frac{\partial}{\partial y} \hat{K}_y = H_0 k_y \sin k_y y \left[ \frac{k_z^2}{k_x^2 + k_y^2} + 1 \right]$$

$$\nabla_{\Sigma} \cdot \bar{K} + \frac{\partial \sigma_f}{\partial t} = H_0 k_y \sin k_y y \left[ \frac{k_z^2}{k_x^2 + k_y^2} + 1 - \frac{k^2}{k_x^2 + k_y^2} \right] = 0$$

30. a)  $f_c = \frac{\omega_c}{2\pi} = \frac{c}{2} \left[ \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right]^{1/2}$

$$a = b \rightarrow f_c = \frac{c}{2a} [m^2 + n^2]^{1/2}$$

$$c = 3 \times 10^8, a = 10^{-2} \rightarrow f_c = 1.5 \times 10^{10} [m^2 + n^2]^{1/2}$$

m	n	$f_c$
1	0	$1.50 \times 10^{10}$
1	1	$2.12 \times 10^{10}$
2	0	$3 \times 10^{10}$
2	1	$3.35 \times 10^{10}$
2	2	$4.24 \times 10^{10}$
3	0	$4.5 \times 10^{10}$
3	1	$4.74 \times 10^{10}$
3	2	$5.41 \times 10^{10}$
4	0	$6.0 \times 10^{10}$
3	3	$6.36 \times 10^{10}$

b)  $f_{co} = \frac{c}{2a} \rightarrow a = \frac{c}{2f_{co}} = \frac{1.5 \times 10^8}{f_{co}}$

GUIDED ELECTROMAGNETIC WAVES

$f_{co}$	$a$
$10^{10}$	1.5 cm
$10^8$	1.5 m
$10^6$	150 m
$10^4$	15 km
$10^2$	$1.5 \times 10^6$ m

31. a) TM

$$\hat{E}_z(x, y, z) = E_o \sin k_x x \sin k_y y \sin k_z z; k_x = m\pi/a$$

$$\hat{E}_y(x, y, z) = \frac{E_o k_x k_z}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \cos k_z z; k_y = n\pi/b$$

$$\hat{E}_x(x, y, z) = \frac{E_o k_x k_z}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \cos k_z z; k_z = p\pi/\ell$$

$$\hat{H}_x = -\frac{1}{j\omega\mu} \left( \frac{\partial \hat{E}_z}{\partial y} - \frac{\partial \hat{E}_y}{\partial z} \right)$$

$$= -\frac{E_o}{j\omega\mu} \sin k_x x \cos k_y y \sin k_z z \left( k_y + \frac{k_x k_z^2}{k_x^2 + k_y^2} \right)$$

$$= -\frac{E_o k_y}{j\omega\mu} \frac{k^2}{(k_x^2 + k_y^2)} \sin k_x x \cos k_y y \sin k_z z$$

$$\hat{H}_y = -\frac{1}{j\omega\mu} \left( \frac{\partial \hat{E}_x}{\partial z} - \frac{\partial \hat{E}_z}{\partial x} \right)$$

$$= \frac{E_o}{j\omega\mu} \cos k_x x \sin k_y y \sin k_z z \left( \frac{k_x k_z^2}{k_x^2 + k_y^2} + k_x \right)$$

$$= \frac{E_o k_x}{j\omega\mu} \frac{k^2}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \sin k_z z$$

$$\hat{H}_z = -\frac{1}{j\omega\mu} \left( \frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} \right) = 0$$

TE

$$\hat{H}_z(x, y, z) = H_o \cos k_x x \cos k_y y \sin k_z z$$

$$\hat{H}_y(x, y, z) = \frac{-H_0 k_y z}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \cos k_z z$$

$$\hat{H}_x(x, y, z) = \frac{-H_0 k_x z}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \cos k_z z$$

$$\begin{aligned} \hat{E}_x &= \frac{1}{j\omega\epsilon} \left( \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z} \right) \\ &= \frac{-H_0}{j\omega\epsilon} \cos k_x x \sin k_y y \sin k_z z \left( k_y + \frac{k_y k_z^2}{k_x^2 + k_y^2} \right) \\ &= \frac{-H_0 k_y^2 k_z}{j\omega\epsilon (k_x^2 + k_y^2)} \cos k_x x \sin k_y y \sin k_z z \end{aligned}$$

$$\begin{aligned} \hat{E}_y &= \frac{1}{j\omega\epsilon} \left( \frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x} \right) \\ &= \frac{H_0}{j\omega\epsilon} \sin k_x x \cos k_y y \sin k_z z \left( \frac{k_x k_z^2}{k_x^2 + k_y^2} + k_x \right) \\ &= \frac{H_0 k_x^2 k_z}{j\omega\epsilon (k_x^2 + k_y^2)} \sin k_x x \cos k_y y \sin k_z z \end{aligned}$$

$$\hat{E}_z = \frac{1}{j\omega\epsilon} \left( \frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} \right) = 0$$

$$b) \quad k^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2 = \frac{\omega^2}{c^2}$$

$$c) \quad TE_{101}$$

$$\hat{E} = \frac{j\omega\mu H_0}{k_x} \sin k_x x \sin k_z z \bar{i}_y; \quad k_x = \frac{\pi}{a}, \quad k_y = 0, \quad k_z = \frac{\pi}{\ell}$$

$$\hat{H} = H_0 \left[ -\frac{k_z}{k_x} \sin k_x x \cos k_z z \bar{i}_x + \cos k_x x \sin k_z z \bar{i}_z \right]$$

$$\begin{aligned} \hat{K}(x, y=0, z) &= -\hat{K}(x, y=b, z) = -\hat{H}_x(x, y=0, z) \bar{i}_z + \hat{H}_z(x, y=0, z) \bar{i}_x \\ &= H_0 \left[ \frac{k_z}{k_x} \sin k_x x \cos k_z z \bar{i}_z + \cos k_x x \sin k_z z \bar{i}_x \right] \end{aligned}$$



$$\begin{aligned}\hat{K}(x=0, y, z) &= \hat{K}(x=a, y, z) = -\hat{H}_z(x=0, y, z)\hat{i}_y \\ &= -H_0 \sin k_z z\end{aligned}$$

$$\begin{aligned}\langle P_{dA}(x, y=0, z) \rangle &= \langle P_{dA}(x, y=b, z) \rangle = \frac{1}{2} \operatorname{Re}(\hat{E} \cdot \hat{K}^*) \\ &= \frac{1}{2} \frac{H_0^2}{\sigma_w \delta} \left[ \left( \frac{k_z}{k_x} \right)^2 \sin^2 k_x x \cos^2 k_z z + \cos^2 k_x x \sin^2 k_z z \right]\end{aligned}$$

$$\langle P_{dA}(x=0, y, z) \rangle = \langle P_{dA}(x=a, y, z) \rangle = \frac{1}{2} \frac{H_0^2}{\sigma_w \delta} \sin^2 k_z z$$

$$\begin{aligned}\langle P_d \rangle &= \int_{x=0}^a \int_{z=0}^{\ell} 2 \langle P_{dA}(x, y=0, z) \rangle dx dz + \int_{y=0}^b \int_{z=0}^{\ell} 2 \langle P_{dA}(x=0, y, z) \rangle dy dz \\ &= \frac{H_0^2}{\sigma_w \delta} \left\{ \left( \frac{a}{\ell} \right)^2 \frac{a}{2} \frac{\ell}{2} + \frac{a}{2} \frac{\ell}{2} + \frac{b\ell}{2} \right\} \\ &= \frac{H_0^2 \ell}{2 \sigma_w \delta} \left\{ \frac{a}{2} \left( 1 + \left( \frac{a}{\ell} \right)^2 \right) + b \right\}\end{aligned}$$

$$\begin{aligned}d) \quad \langle w \rangle &= \frac{1}{2} \epsilon |\vec{E}|^2 + \frac{1}{2} \mu |\vec{H}|^2 \\ &= \frac{\omega^2 \mu^2 H_0^2}{k_x^2} \sin^2 k_x x \sin^2 k_z z + \frac{1}{2} \mu H_0^2 \left[ \left( \frac{k_z}{k_x} \right)^2 \sin^2 k_x x \cos^2 k_z z + \cos^2 k_x x \sin^2 k_z z \right] \\ &= \frac{1}{2} \mu H_0^2 \left\{ \frac{\omega^2 \epsilon \mu}{k_x^2} \sin^2 k_x x \sin^2 k_z z + \left( \frac{a}{\ell} \right)^2 \sin^2 k_x x \cos^2 k_z z + \cos^2 k_x x \sin^2 k_z z \right\}\end{aligned}$$

$$\begin{aligned}\langle W \rangle &= \int_{x=0}^a \int_{y=0}^b \int_{z=0}^{\ell} \langle w \rangle dx dy dz \\ &= \frac{1}{2} \mu H_0^2 \left\{ \frac{\omega^2 \epsilon \mu a^2}{\pi^2} \frac{a}{2} \frac{\ell b}{2} + \left( \frac{a}{\ell} \right)^2 \frac{a}{2} \frac{\ell b}{2} + \frac{a}{2} \frac{\ell b}{2} \right\} \\ &= \frac{1}{8} \mu H_0^2 a \ell b \left\{ \frac{\omega^2 \epsilon \mu a^2}{\pi^2} + \left( \frac{a}{\ell} \right)^2 + 1 \right\}\end{aligned}$$

$$e) \quad Q = \frac{\omega_0 \langle W \rangle}{\langle P_d \rangle} = \frac{\omega_0 a b \left[ \frac{\omega_0^2 \epsilon \mu a^2}{\pi^2} + \left( \frac{a}{\ell} \right)^2 + 1 \right] \mu \sigma_w \delta}{\left[ \frac{a}{2} \left( 1 + \left( \frac{a}{\ell} \right)^2 \right) + b \right]}$$

$$= \frac{2ab}{\delta} \frac{\left[ \frac{\omega_o^2 \epsilon \mu a^2}{\pi^2} + \left(\frac{a}{\ell}\right)^2 + 1 \right]}{\left[ \frac{a}{2} \left( 1 + \left(\frac{a}{\ell}\right)^2 \right) + b \right]}$$

$$\delta = \sqrt{\frac{2}{\omega_o \mu \sigma_w}}, \quad \omega_o^2 = \pi^2 c^2 \left[ \frac{1}{a^2} + \frac{1}{\ell^2} \right]$$

$$Q = \frac{4ab}{\delta} \frac{1}{\left[ \frac{a}{2} + \frac{b \pi^2 c^2}{\omega_o^2 a^2} \right]}$$

### Section 8.7

32. a)  $k_x^2 + k_z^2 = \omega^2 \epsilon \mu$

$$-\alpha^2 + k_z^2 = \omega^2 \epsilon_o \mu_o$$

$$\alpha = 0 \rightarrow \omega^2 = \frac{k_x^2}{\epsilon \mu - \epsilon_o \mu_o}$$

#### TM

$$\text{Odd: } \alpha = \frac{\epsilon_o}{\epsilon} k_x \tan k_x d = 0 \rightarrow k_x = \frac{n\pi}{d}$$

$$\text{Even: } \alpha = -\frac{\epsilon_o}{\epsilon} k_x \cot k_x d = 0 \rightarrow k_x = \frac{(2n+1)\pi}{2d}$$

#### TE

$$\text{Odd: } \alpha = \frac{\mu_o}{\mu} k_x \tan k_x d = 0 \rightarrow k_x = \frac{n\pi}{d}$$

$$\text{Even: } \alpha = -\frac{\mu_o}{\mu} k_x \cot k_x d = 0 \rightarrow k_x = \frac{(2n+1)\pi}{2d}$$

b)  $k_x d \ll 1 \rightarrow \tan k_x d \approx k_x d, \cot k_x d \approx \frac{1}{k_x d}$

#### TM

$$\text{Odd: } \alpha \approx \frac{\epsilon_o}{\epsilon} k_x^2 d, \quad k_x^2 + \alpha^2 = \omega^2 (\epsilon \mu - \epsilon_o \mu_o)$$

$$k_x^4 + k_x^2 \left( \frac{\epsilon_o}{\epsilon d} \right)^2 - \omega^2 (\epsilon \mu - \epsilon_o \mu_o) \left( \frac{\epsilon_o}{\epsilon d} \right)^2 = 0$$

$$k_x^2 = -\frac{1}{2} \left( \frac{\epsilon}{\epsilon_o d} \right)^2 \pm \sqrt{\frac{1}{4} \left( \frac{\epsilon}{\epsilon_o d} \right)^4 + \omega^2 \left( \frac{\epsilon}{\epsilon_o d} \right)^2 (\epsilon\mu - \epsilon_o \mu_o)}$$

$$k_z^2 = \omega^2 \epsilon_o \mu_o + \alpha^2 = \omega^2 \epsilon\mu - k_x^2$$

$$\text{Even: } \alpha \approx -\frac{\epsilon_o}{\epsilon d}, \quad k_x^2 = -\alpha^2 + \omega^2 (\epsilon\mu - \epsilon_o \mu_o) = \omega^2 (\epsilon\mu - \epsilon_o \mu_o) - \left( \frac{\epsilon_o}{\epsilon d} \right)^2$$

TE

$$\text{Odd: } \alpha \approx \frac{\mu_o}{\mu d}, \quad k_x^2 = -\frac{1}{2} \left( \frac{\mu}{\mu_o d} \right)^2 \pm \sqrt{\frac{1}{4} \left( \frac{\mu}{\mu_o d} \right)^4 + \omega^2 \left( \frac{\mu}{\mu_o d} \right)^2 (\epsilon\mu - \epsilon_o \mu_o)}$$

$$k_z^2 = \omega^2 \epsilon_o \mu_o + \alpha^2 = \omega^2 \epsilon\mu - k_x^2$$

$$\text{Even: } \alpha = -\frac{\mu_o}{\mu d}, \quad k_x^2 = -\alpha^2 + \omega^2 (\epsilon\mu - \epsilon_o \mu_o) = \omega^2 (\epsilon\mu - \epsilon_o \mu_o) - \left( \frac{\mu_o}{\mu d} \right)^2$$

c) TM

$$\text{Odd: } \langle \vec{S} \rangle = \frac{1}{2} \text{Re} \hat{\vec{E}} \times \hat{\vec{H}}^* = \frac{1}{2} \text{Re} [(\hat{E}_x \bar{i}_x + \hat{E}_z \bar{i}_z) \times (\hat{H}_y^* \bar{i}_y)]$$

$$= \frac{1}{2} \text{Re} [\hat{E}_x \hat{H}_y^* \bar{i}_z - \hat{E}_z \hat{H}_y^* \bar{i}_x]$$

$$= \begin{cases} \frac{\omega \epsilon_o k_z}{\alpha^2} |A_2|^2 e^{-2\alpha(x-d)} \bar{i}_z & x > d \\ \frac{\omega \epsilon k_z}{k_x^2} |A_1|^2 \cos^2 k_x x \bar{i}_z & |x| < d \\ \frac{\omega \epsilon_o k_z}{\alpha^2} |A_3|^2 e^{2\alpha(x+d)} \bar{i}_z & x < -d \end{cases}$$

$$\langle P_z \rangle = \int_{-\infty}^{-d} \frac{\omega \epsilon_o k_z}{\alpha^2} |A_3|^2 e^{2\alpha(x+d)} dx + \int_{-d}^d \frac{\omega \epsilon k_z}{k_x^2} |A_1|^2 \cos^2 k_x x dx$$

$$+ \int_d^{\infty} \frac{\omega \epsilon_o k_z}{\alpha^2} |A_2|^2 e^{-2\alpha(x-d)} dx$$

$$= \frac{\omega \epsilon_0 k_z}{2\alpha^3} [|A_3|^2 + |A_2|^2] + \frac{\omega \epsilon k_z}{k_x^3} |A_1|^2 [k_x d + \frac{1}{2} \sin 2k_x d]$$

$$|A_3| = |A_2| = |A_1| \sin k_x d = |A_1| \frac{\epsilon \alpha}{\epsilon_0 k_x} \cos k_x d$$

$$\langle P_z \rangle = |A_1|^2 \left[ \frac{\omega \epsilon_0 k_z}{\alpha^3} \sin^2 k_x d + \frac{\omega \epsilon k_z}{k_x^3} (k_x d + \frac{1}{2} \sin 2k_x d) \right]$$

$$\text{Even: } \langle \bar{S} \rangle = \frac{1}{2} \text{Re} [\hat{\vec{E}} \times \hat{\vec{H}}^*] = \frac{1}{2} \text{Re} [(\hat{E}_x \hat{i}_x + \hat{E}_z \hat{i}_z) \times \hat{H}_y^* \hat{i}_y]$$

$$= \frac{1}{2} \text{Re} [\hat{E}_x \hat{H}_y^* \hat{i}_z - \hat{E}_z \hat{H}_y^* \hat{i}_x]$$

$$= \begin{cases} \frac{\omega \epsilon_0 k_z}{\alpha^2} |B_2|^2 e^{-2\alpha(x-d)} \hat{i}_z & x > d \\ \frac{\omega \epsilon k_z}{k_x^2} |B_1|^2 \sin^2 k_x x \hat{i}_z & -d < x < d \\ \frac{\omega \epsilon_0 k_z}{\alpha^2} |B_3|^2 e^{2\alpha(x-d)} \hat{i}_z & x < -d \end{cases}$$

$$\langle P_z \rangle = \frac{\omega \epsilon_0 k_z}{2\alpha^3} [|B_3|^2 + |B_2|^2] + \frac{\omega \epsilon k_z}{k_x^3} |B_1|^2 [k_x d - \frac{1}{2} \sin 2k_x d]$$

$$|B_2| = |B_3| = |B_1| \cos k_x d = \frac{|B_1| \epsilon \alpha}{\epsilon_0 k_x} \sin k_x d$$

$$\langle P_z \rangle = |B_1|^2 \omega k_z \left[ \frac{\epsilon_0}{\alpha^3} \cos^2 k_x d + \frac{\epsilon}{k_x^3} (k_x d - \frac{1}{2} \sin 2k_x d) \right]$$

TE

$$\text{Odd: } \langle \bar{S} \rangle = \frac{1}{2} \text{Re} (\hat{\vec{E}} \times \hat{\vec{H}}^*) = \frac{1}{2} \text{Re} [\hat{E}_y \hat{i}_y \times (\hat{H}_x^* \hat{i}_x + \hat{H}_z^* \hat{i}_z)]$$

$$= \frac{1}{2} \text{Re} [-\hat{E}_y \hat{H}_x^* \hat{i}_z + \hat{E}_y \hat{H}_z^* \hat{i}_x]$$

$$= \begin{cases} \frac{\omega \mu_o k_z}{\alpha^2} |A_2|^2 e^{-2\alpha(x-d)} \bar{i}_z & x > d \\ \frac{\omega \mu k_z}{k_x^2} |A_1|^2 \cos^2 k_x x \bar{i}_z & -d < x < d \\ \frac{\omega \mu_o k_z}{\alpha^2} |A_3|^2 e^{2\alpha(x-d)} \bar{i}_z & x < -d \end{cases}$$

$$\langle P_z \rangle = \frac{\omega \mu_o k_z}{2\alpha^3} [|A_2|^2 + |A_3|^2] + \frac{\omega \mu k_z}{k_x^3} |A_1|^2 [k_x d + \frac{1}{2} \sin 2k_x d]$$

$$|A_2| = |A_3| = |A_1| \sin k_x d = \frac{\mu \alpha |A_1| \cos k_x d}{\mu_o k_x}$$

$$\langle P_z \rangle = |A_1|^2 \left[ \frac{\omega \mu_o k_z}{\alpha^3} \sin^2 k_x d + \frac{\omega \mu k_z}{k_x^3} [k_x d + \frac{1}{2} \sin 2k_x d] \right]$$

$$\text{Even: } \langle \bar{S} \rangle = \begin{cases} \frac{\omega \mu_o k_z}{\alpha^2} |B_2|^2 e^{-2\alpha(x-d)} \bar{i}_z & x > d \\ \frac{\omega \mu k_z}{k_x^2} |B_1|^2 \sin^2 k_x x \bar{i}_z & -d < x < d \\ \frac{\omega \mu_o k_z}{\alpha^2} |B_3|^2 e^{2\alpha(x+d)} \bar{i}_z & x < -d \end{cases}$$

$$\langle P_z \rangle = \frac{\omega \mu_o k_z}{2\alpha^3} [|B_2|^2 + |B_3|^2] + \frac{\omega \mu k_z}{k_x^3} |B_1|^2 [k_x d - \frac{1}{2} \sin 2k_x d]$$

$$|B_2| = |B_3| = |B_1| \cos k_x d = \frac{\mu \alpha}{\mu_o k_x} |B_1| \sin k_x d$$

$$\langle P_z \rangle = |B_1|^2 \left[ \frac{\omega \mu_o k_z}{\alpha^3} \cos^2 k_x d + \frac{\omega \mu k_z}{k_x^3} [k_x d - \frac{1}{2} \sin 2k_x d] \right]$$

$$d) \quad \langle P_d \rangle = \frac{1}{2} \sigma |\hat{E}|^2 \quad -d < x < d$$

$$\alpha_z = \frac{1}{2} \frac{\langle P_d \rangle}{\langle P_z \rangle}; \quad \langle P_d \rangle = \int_{-d}^{+d} \frac{1}{2} \sigma |\hat{E}|^2 dx$$

TM

$$\text{Odd: } \langle p_d \rangle = \frac{1}{2} \sigma |A_1|^2 \left[ \sin^2 k_x x + \left( \frac{k_z}{k_x} \right)^2 \cos^2 k_x x \right]$$

$$\langle P_d \rangle = \frac{1}{2} \sigma \frac{|A_1|^2}{k_x} \left[ k_x d \left( 1 + \left( \frac{k_z}{k_x} \right)^2 \right) - \frac{1}{2} \sin 2k_x d \left( 1 - \left( \frac{k_z}{k_x} \right)^2 \right) \right]$$

$$\alpha_z = \frac{\sigma}{4\omega k_z k_x} \frac{\left[ d \left( 1 + \left( \frac{k_z}{k_x} \right)^2 \right) - \frac{1}{2} \sin 2k_x d \left( 1 - \left( \frac{k_z}{k_x} \right)^2 \right) \right]}{\left[ \frac{\epsilon_0}{3} \sin^2 k_x d + \frac{\epsilon_3}{k_x} \left( k_x d + \frac{1}{2} \sin 2k_x d \right) \right]}$$

$$\text{Even: } \langle p_d \rangle = \frac{1}{2} \sigma |B_1|^2 \left[ \cos^2 k_x x + \left( \frac{k_z}{k_x} \right)^2 \sin^2 k_x x \right]$$

$$\langle P_d \rangle = \frac{1}{2} \sigma \frac{|B_1|^2}{k_x} \left[ k_x d \left( 1 + \left( \frac{k_z}{k_x} \right)^2 \right) + \frac{1}{2} \sin 2k_x d \left( 1 - \left( \frac{k_z}{k_x} \right)^2 \right) \right]$$

TE

$$\text{Odd: } \langle p_d \rangle = \frac{\sigma}{2} \left( \frac{\omega \mu}{k_x} \right)^2 |A_1|^2 \cos^2 k_x x$$

$$\langle P_d \rangle = \frac{\sigma}{2} \left( \frac{\omega \mu}{k_x} \right)^2 \frac{|A_1|^2}{k_x} \left[ k_x d + \frac{1}{2} \sin 2k_x d \right]$$

$$\text{Even: } \langle p_d \rangle = \frac{\sigma}{2} \left( \frac{\omega \mu}{k_x} \right)^2 |B_1|^2 \sin^2 k_x x$$

$$\langle P_d \rangle = \frac{\sigma}{2} \left( \frac{\omega \mu}{k_x} \right)^2 \frac{|B_1|^2}{k_x} \left[ k_x d - \frac{1}{2} \sin 2k_x d \right]$$

33. a)  $E_y(x=0) = E_z(x=0) = 0$

TM even not allowed

TE odd not allowed

Allowed modes: TM odd, TE even

b) TM odd

$$\hat{\sigma}_f(x=0) = \epsilon \hat{E}_x(x=0) = - \frac{j \epsilon k_z A_1}{k_x}$$

$$\hat{K}_z(x=0) = \hat{H}_y(x=0) = -\frac{j\omega\epsilon A_1}{k_x}$$

TE even

$$\hat{\sigma}_f(x=0) = 0$$

$$\hat{K}_y(x=0) = B_1$$

$$c) \text{ TM odd: } \nabla_{\Sigma} \cdot \bar{K} + \frac{\partial \sigma_f}{\partial t} \rightarrow -jk_x \hat{K}_z + j\omega \hat{\sigma}_f = -jk_z \left( \frac{-j\omega\epsilon A_1}{k_x} \right) + j\omega \left( \frac{-j\epsilon k_z A_1}{k_x} \right) = 0$$

$$\text{TE even: } \nabla_{\Sigma} \cdot \bar{K} = 0$$

$$d) \langle P_d \rangle = \frac{|\bar{K}|^2}{\sigma_w \delta} = \begin{cases} \frac{|\frac{\omega\epsilon A_1}{k_x}|^2}{\sigma_w \delta} & \text{TM} \\ \frac{|B_1|^2}{\sigma_w \delta} & \text{TE} \end{cases}$$

$$e) \alpha_z = \frac{1}{2} \frac{\langle P_d \rangle}{\langle P_z \rangle}$$

$$\text{TM odd: } \langle S_z \rangle = \begin{cases} \frac{\omega\epsilon_o k_z}{\alpha^2} |A_2|^2 e^{-2\alpha(x-d)} = \frac{\omega\epsilon_o^2 k_z}{\epsilon_o k_x^2} |A_1|^2 \cos^2 k_x d e^{-2\alpha(x-d)} & x > d \\ \frac{\omega\epsilon k_z}{k_x^2} |A_1|^2 \cos^2 k_x x & 0 < x < d \end{cases}$$

$$\text{TE even: } \langle S_z \rangle = \begin{cases} \frac{\omega\mu_o k_z}{\alpha^2} |B_2|^2 e^{-2\alpha(x-d)} = \frac{\omega\mu_o^2 k_z}{\mu_o k_x^2} |B_1|^2 \sin^2 k_x d e^{-2\alpha(x-d)} & x > d \\ \frac{\omega\mu k_z}{k_x^2} |B_1|^2 \sin^2 k_x x & 0 < x < d \end{cases}$$

TM

$$\begin{aligned} \langle P_z \rangle &= \frac{\omega\epsilon k_z}{k_x^2} |A_1|^2 \left\{ \int_0^d \cos^2 k_x x dx + \int_d^\infty \frac{\epsilon}{\epsilon_o} \cos^2 k_x d e^{-2\alpha(x-d)} dx \right\} \\ &= \frac{\omega\epsilon k_z}{k_x^3} |A_1|^2 \left\{ \frac{1}{2} k_x d + \frac{1}{4} \sin 2k_x d + \frac{\epsilon}{\epsilon_o} \frac{\cos^2 k_x d}{2\alpha} \right\} \end{aligned}$$

$$\alpha_z = \frac{\omega \epsilon k_x}{\sigma_w \delta k_z} \left[ k_x d + \frac{1}{2} \sin 2k_x d + \frac{\epsilon}{\epsilon_0 \alpha} \cos^2 k_x d \right]^{-1}$$

TE

$$\langle P_z \rangle = \frac{\omega \mu k_x}{k_x^2} |B_1|^2 \left\{ \int_0^d \sin^2 k_x x dx + \int_d^\infty \frac{\mu}{\mu_0} \sin^2 k_x d e^{-2\alpha(x-d)} dx \right\}$$

$$= \frac{\omega \mu k_x}{k_x^3} |B_1|^2 \left\{ \frac{1}{2} k_x d - \frac{1}{4} \sin 2k_x d + \frac{\mu}{\mu_0} \frac{\sin^2 k_x d}{2\alpha} \right\}$$

$$\alpha_z = \frac{k_x^3}{\sigma_w \delta \omega \mu k_z} \left[ k_x d - \frac{1}{2} \sin 2k_x d + \frac{\mu}{\mu_0 \alpha} \sin^2 k_x d \right]^{-1}$$



# CHAPTER 9

## RADIATION

### Section 9.1

$$1. \quad a) \quad \nabla \times \bar{\mathbf{E}} = - \frac{\partial \bar{\mathbf{B}}}{\partial t}$$

$$\nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}}_s + \sigma \bar{\mathbf{E}} + \epsilon \frac{\partial \bar{\mathbf{E}}}{\partial t}$$

$$\nabla \cdot \bar{\mathbf{B}} = 0 \rightarrow \bar{\mathbf{B}} = \nabla \times \bar{\mathbf{A}}$$

$$\nabla \cdot \bar{\mathbf{E}} = \frac{\rho}{\epsilon}$$

$$b) \quad \nabla \times (\bar{\mathbf{E}} + \frac{\partial \bar{\mathbf{A}}}{\partial t}) = 0 \rightarrow \bar{\mathbf{E}} = - \frac{\partial \bar{\mathbf{A}}}{\partial t} - \nabla V$$

$$\nabla \times \bar{\mathbf{H}} = \frac{1}{\mu} \nabla \times (\nabla \times \bar{\mathbf{A}}) = \frac{1}{\mu} [\nabla (\nabla \cdot \bar{\mathbf{A}}) - \nabla^2 \bar{\mathbf{A}}] = \bar{\mathbf{J}}_s - \sigma [\frac{\partial \bar{\mathbf{A}}}{\partial t} + \nabla V] - \epsilon \frac{\partial^2 \bar{\mathbf{A}}}{\partial t^2} - \epsilon \nabla \frac{\partial V}{\partial t}$$

$$\nabla^2 \bar{\mathbf{A}} - \nabla [\nabla \cdot \bar{\mathbf{A}} + \mu \sigma V + \mu \epsilon \frac{\partial V}{\partial t}] = \mu \sigma \frac{\partial \bar{\mathbf{A}}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \bar{\mathbf{A}}}{\partial t^2} = -\mu \bar{\mathbf{J}}_s$$

$$\text{Gauge condition: } \nabla \cdot \bar{\mathbf{A}} + \mu \sigma V + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$$

$$\nabla^2 \bar{\mathbf{A}} - \mu \sigma \frac{\partial \bar{\mathbf{A}}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \bar{\mathbf{A}}}{\partial t^2} = -\mu \bar{\mathbf{J}}_s$$

$$\nabla \cdot \bar{\mathbf{E}} = - \frac{\partial}{\partial t} \nabla \cdot \bar{\mathbf{A}} - \nabla^2 V = \frac{\rho}{\epsilon}$$

$$\nabla^2 V - \mu \sigma \frac{\partial V}{\partial t} - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = - \frac{\rho}{\epsilon}$$

$$c) \quad V = \text{Re } \hat{V}(r) e^{j\omega t}, \quad \rho = 0$$

$$\frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d\hat{V}}{dr}) - j\omega\mu\sigma \hat{V} + \frac{\omega^2}{c^2} \hat{V} = 0$$

$$\frac{d^2}{dr^2} (r\hat{V}) + (\frac{\omega^2}{c^2} - j\omega\mu\sigma) r\hat{V} = 0$$

$$\hat{V} = \frac{\hat{Q}}{4\pi\epsilon r} e^{-j\sqrt{\frac{\omega^2}{c^2} - \omega_p^2} r}$$

$$d) \quad \nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \frac{\partial \bar{H}}{\partial t} = \frac{\partial \bar{J}_s}{\partial t} + \omega_p^2 \epsilon \bar{E} + \epsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\nabla \cdot \bar{B} = 0 \rightarrow \bar{B} = \nabla \times \bar{A}$$

$$\bar{E} = - \frac{\partial \bar{A}}{\partial t} - \nabla V$$

$$\nabla \times (\nabla \times \frac{\partial \bar{A}}{\partial t}) = \nabla (\nabla \cdot \frac{\partial \bar{A}}{\partial t}) - \nabla^2 \frac{\partial \bar{A}}{\partial t} = \mu \frac{\partial \bar{J}_s}{\partial t} - \frac{\omega_p^2}{c^2} (\frac{\partial \bar{A}}{\partial t} + \nabla V) - \frac{1}{c^2} \frac{\partial^3 \bar{A}}{\partial t^3} - \frac{1}{c^2} \nabla \frac{\partial^2 V}{\partial t^2}$$

$$\nabla^2 \frac{\partial \bar{A}}{\partial t} - \nabla [\nabla \cdot \frac{\partial \bar{A}}{\partial t} + \frac{\omega_p^2}{c^2} V + \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}] - \frac{\omega_p^2}{c^2} \frac{\partial \bar{A}}{\partial t} - \frac{1}{c^2} \frac{\partial^3 \bar{A}}{\partial t^3} = -\mu \frac{\partial \bar{J}_s}{\partial t}$$

$$\text{Gauge condition: } \nabla \cdot \frac{\partial \bar{A}}{\partial t} + \frac{\omega_p^2}{c^2} V + \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0$$

$$\nabla^2 \bar{A} - \frac{\omega_p^2}{c^2} \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}_s$$

$$\nabla \cdot \bar{E} = - \nabla \cdot \frac{\partial \bar{A}}{\partial t} - \nabla^2 V = \frac{\rho}{\epsilon}$$

$$\nabla^2 V - \frac{\omega_p^2}{c^2} V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = - \frac{\rho}{\epsilon}$$

$$\text{For } \rho = 0 \text{ with } Q(r=0, t) = \text{Re} \hat{Q} e^{j\omega t}$$

$$\frac{d^2}{dr^2} (r\hat{V}) + \frac{(\omega^2 - \omega_p^2)}{c^2} r\hat{V} = 0$$

$$\hat{V} = \frac{\hat{Q}}{4\pi\epsilon r} e^{-j\frac{r}{c}\sqrt{\omega^2 - \omega_p^2}}$$

$$2. \quad a) \quad \nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}_f = 0$$

$$\bar{A} = \text{Re} \hat{A}_x e^{j(\omega t - k_x x - k_z z)} \hat{i}_x \quad z > 0$$

# RADIATION

$$\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial z^2} + \frac{\omega^2}{c^2} A_x = 0 \rightarrow k_x^2 + k_z^2 = \frac{\omega^2}{c^2}$$

$$\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \rightarrow -jk_x \hat{A}_x = -\frac{j\omega}{c^2} \hat{V} \rightarrow \hat{V} = \frac{k_x c^2}{\omega} \hat{A}_x$$

$$\begin{aligned} \text{(b)} \quad \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \nabla V \rightarrow \hat{E} = -j\omega \hat{A}_x \hat{i}_x + (jk_x \hat{i}_x + jk_z \hat{i}_z) \hat{V} \\ &= \hat{i}_x j \left( \frac{k_x^2 c^2 - \omega^2}{\omega} \right) \hat{A}_x + \hat{i}_z \frac{jk_z k_x c^2}{\omega} \hat{A}_x \end{aligned}$$

$$\vec{B} = \nabla \times \vec{A} = \hat{i}_y \frac{\partial A_x}{\partial z} \rightarrow \hat{B}_y = -jk_z \hat{A}_x$$

$$\hat{H}_y = -\frac{jk_z}{\mu} \hat{A}_x \rightarrow \hat{H}_y(z=0) = -\frac{K_o}{2} = -\frac{jk_z}{\mu} \hat{A}_x \rightarrow \hat{A}_x = \frac{\mu K_o}{2jk_z}$$

For  $z < 0$  let  $k_z \rightarrow -k_z$ .

$$\text{(c)} \quad \nabla^2 A_x - \frac{1}{c^2} \frac{\partial^2 A_x}{\partial t^2} = -\mu J_o e^{j(\omega t - k_x x)} \quad |z| < a$$

$$A_x = \text{Re} \hat{A}_x(z) e^{j(\omega t - k_x x)}$$

$$\frac{d^2 \hat{A}_x}{dz^2} - \left( k_x^2 - \frac{\omega^2}{c^2} \right) \hat{A}_x = -\mu J_o$$

$$\hat{A}_x = \text{Re} \left\{ \underbrace{\left( \frac{\mu J_o}{k_x^2 - \frac{\omega^2}{c^2}} \right) e^{j(\omega t - k_x x)}}_{\text{Particular Solution}} + \underbrace{A_1 e^{j(\omega t - k_x x - k_z z)} + A_2 e^{j(\omega t - k_x x + k_z z)}}_{\text{Homogeneous Solution}} \right\}$$

$$\hat{A}_x = \begin{cases} \frac{\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x - k_z(z-z'))} & z > z' \\ -\frac{\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x + k_z(z-z'))} & z < z' \end{cases}$$

# RADIATION

$$z > a$$

$$\begin{aligned}\hat{A}_x &= \int_{-a}^a \frac{\mu J_0 dz'}{2jk_z} e^{j(\omega t - k_x x - k_z(z-z'))} \\ &= \frac{\mu J_0}{2k_z^2} e^{j(\omega t - k_x x - k_z z)} [e^{jk_z a} - e^{-jk_z a}] \\ &= \frac{\mu J_0 j}{k_z^2} \text{sinc}_z a e^{j(\omega t - k_x x - k_z z)}\end{aligned}$$

$$z < -a$$

$$\begin{aligned}\hat{A}_x &= \int_{-a}^a \frac{-\mu J_0 dz'}{2jk_z} e^{j(\omega t - k_x x + k_z(z-z'))} \\ &= \frac{\mu J_0}{k_z^2} e^{j(\omega t - k_x x + k_z z)} [e^{-jk_z a} - e^{jk_z a}] \\ &= \frac{-\mu J_0 j}{k_z^2} \text{sinc}_z a e^{j(\omega t - k_x x + k_z z)}\end{aligned}$$

$$-a < z < a$$

$$\begin{aligned}\hat{A}_x &= \int_{-a}^z \frac{-\mu J_0 dz'}{2jk_z} e^{j(\omega t - k_x x - k_z(z-z'))} + \int_z^a \frac{\mu J_0 dz'}{2jk_z} e^{j(\omega t - k_x x + k_z(z-z'))} \\ &= \frac{\mu J_0}{2k_z^2} e^{j(\omega t - k_x x)} [1 - e^{-jk_z(z+a)} + e^{jk_z(z-a)} - 1] \\ &= \frac{j\mu J_0}{k_z^2} e^{-jk_z a} \text{sinc}_z z e^{j(\omega t - k_x x)}\end{aligned}$$

$$A_x = \begin{cases} \text{Re} \left[ \frac{\mu J_0 j}{k_z^2} \text{sinc}_z a e^{j(\omega t - k_x x - k_z z)} \right] & z > a \\ \text{Re} \left[ \frac{\mu J_0 j}{k_z^2} e^{-jk_z a} \text{sinc}_z z e^{j(\omega t - k_x x)} \right] & |z| < a \\ \text{Re} \left[ \frac{-\mu J_0 j}{k_z^2} \text{sinc}_z a e^{j(\omega t - k_x x + k_z z)} \right] & z < -a \end{cases}$$

$$V = \frac{k_x c^2}{\omega} A_x$$

# RADIATION

$$\vec{E} = - \frac{\partial \vec{A}}{\partial t} - \nabla V$$

$$\hat{E}_x = -j\omega \hat{A}_x + jk_x \hat{V} = \hat{A}_x \left( -j\omega + \frac{jk_x^2 c^2}{\omega} \right) = \frac{j\hat{A}_x}{\omega} (k_x^2 c^2 - \omega^2)$$

$$\hat{E}_z = jk_z \hat{V} = \frac{jk_z k_x c^2}{\omega} \hat{A}_x$$

$$H_y = \frac{1}{\mu} \frac{\partial A_x}{\partial z} \rightarrow \hat{H}_y = \frac{-jk_z}{\mu} \hat{A}_x$$

3. (a)

$$V = \int_S \frac{\sigma_f(t - \frac{r_{QP}}{c})}{4\pi\epsilon r_{QP}} R^2 \sin\theta d\theta d\phi$$

$$r_{QP}^2 = [r^2 + R^2 - 2rR\cos\theta]$$

$$\hat{V} = \frac{\sigma_o R^2}{2\epsilon} \int_{\theta=0}^{\pi} \frac{e^{-j\frac{\omega}{c} r_{QP}}}{r_{QP}} \sin\theta d\theta$$

$$2r_{QP} dr_{QP} = 2rR \sin\theta d\theta \rightarrow \frac{\sin\theta d\theta}{r_{QP}} = \frac{dr_{QP}}{rR}$$

$$\begin{aligned} \hat{V} &= \frac{\sigma_o R}{2\epsilon r} \int_{|r-R|}^{r+R} e^{-j\frac{\omega}{c} r_{QP}} dr_{QP} \\ &= \frac{\sigma_o R}{2\epsilon r \left( \frac{-j\omega}{c} \right)} \left[ e^{-j\frac{\omega}{c} (r+R)} - e^{-j\frac{\omega}{c} |r-R|} \right] \end{aligned}$$

$$r < R$$

$$e^{-j\frac{\omega}{c} (r+R)} - e^{-j\frac{\omega}{c} (R-r)} = e^{-j\frac{\omega}{c} R} \left[ e^{-j\frac{\omega}{c} r} - e^{j\frac{\omega}{c} r} \right]$$

$$\hat{V} = \frac{j\sigma_o R c e^{-j\frac{\omega}{c} R}}{2\epsilon r \omega} \left[ e^{-j\frac{\omega}{c} r} - e^{j\frac{\omega}{c} r} \right]$$

$$r > R$$

$$e^{-j\frac{\omega}{c} (r+R)} - e^{-j\frac{\omega}{c} (r-R)} = e^{-j\frac{\omega}{c} r} \left[ e^{-j\frac{\omega}{c} R} - e^{j\frac{\omega}{c} R} \right]$$

# RADIATION

$$\hat{V} = \frac{j\sigma_o Rce^{-j\frac{\omega}{c}r}}{2\epsilon r\omega} \left[ e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R} \right]$$

$$\nabla \cdot \vec{A} = -\frac{1}{c} \frac{\partial V}{\partial t} \rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r^2 \hat{A}_r) = \frac{-j\omega}{c^2} \hat{V}$$

$$\frac{d}{dr} (r^2 \hat{A}_r) = \frac{\sigma_o Rre^{-j\omega r/c}}{2\epsilon c} \left[ e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R} \right]$$

$$\begin{aligned} \hat{A}_r &= \frac{\sigma_o R}{2\epsilon cr^2} \left[ e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R} \right] \frac{e^{-j\omega r/c}}{\left(\frac{\omega^2}{c^2}\right)} \left(\frac{j\omega r}{c} + 1\right) \\ &= \frac{\sigma_o Rc}{2\epsilon \omega^2 r^2} \left(\frac{j\omega r}{c} + 1\right) e^{-j\omega r/c} \left[ e^{-j\omega R/c} - e^{j\omega R/c} \right] + \frac{C_1}{r^2} \end{aligned}$$

$$r < R$$

$$\frac{d}{dr} (r^2 \hat{A}_r) = \frac{\sigma_o Rr}{2\epsilon c} e^{-j\frac{\omega}{c}R} \left[ e^{-j\frac{\omega}{c}r} - e^{j\frac{\omega}{c}r} \right]$$

$$\hat{A}_r = \frac{\sigma_o Rc}{2\epsilon r^2} \frac{e^{-j\omega R/c}}{\omega^2} \left[ e^{-j\frac{\omega}{c}r} \left(\frac{j\omega r}{c} + 1\right) + e^{j\frac{\omega}{c}r} \left(\frac{j\omega r}{c} - 1\right) \right] + \frac{C_2}{r^2}$$

$$(b) \quad E_r = -\frac{\partial A_r}{\partial t} - \frac{dV}{dr}$$

$$r > R$$

$$\begin{aligned} \hat{E}_r &= \frac{-j\sigma_o Rc}{2\epsilon r^2 \omega} \left(\frac{j\omega r}{c} + 1\right) e^{-j\omega r/c} \left[ e^{-j\omega R/c} - e^{j\omega R/c} \right] [1 - 1] - \frac{j\omega C_1}{r^2} \\ &= \frac{-j\omega C_1}{r^2} \end{aligned}$$

$$r < R$$

$$\begin{aligned} \hat{E}_r &= \frac{-j\sigma_o Rc}{2\epsilon r^2 \omega} e^{-j\omega R/c} \left[ e^{-j\frac{\omega}{c}r} \left(\frac{j\omega r}{c} + 1\right) (1 - 1) + e^{j\omega r/c} \left(\frac{j\omega r}{c} - 1\right) (1 - 1) \right] - \frac{j\omega C_2}{r^2} \\ &= \frac{-j\omega C_2}{r^2} \end{aligned}$$

# RADIATION

$$\hat{E}_r(r=0) = \text{finite} \rightarrow C_2 = 0$$

$$\hat{E}_r(r=R_+) = \frac{\sigma_o}{\epsilon} = \frac{-j\omega C_1}{R^2} \rightarrow C_1 = \frac{j\sigma_o R^2}{\epsilon\omega}$$

$$\hat{E}_r = \begin{cases} \frac{\sigma_o R^2}{\epsilon r^2} & r > R \\ 0 & r < R \end{cases}$$

$$\bar{H} = \frac{1}{\mu} \nabla \times \bar{A} = 0$$

$$(c) \quad \nabla \times \bar{H} = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t} \rightarrow \hat{J}_r = -j\omega\epsilon \hat{E}_r$$

## Section 9.2

$$4. \quad d\ell_{\text{eff}} = \frac{1}{I_o} \int_{-d\ell/2}^{d\ell/2} \hat{I}(z) dz, \quad R = \frac{2\pi\eta}{3} \left( \frac{d\ell_{\text{eff}}}{\lambda} \right)^2$$

$$\hat{\lambda} = -\frac{1}{j\omega} \frac{d\hat{I}}{dz}$$

$$(a) \quad \hat{I}(z) = I_o \cos \alpha z$$

$$d\ell_{\text{eff}} = \frac{d\ell/2}{-d\ell/2} \int_{-d\ell/2}^{d\ell/2} \cos \alpha z dz = \frac{2 \sin \alpha \frac{d\ell}{2}}{\alpha}$$

$$\hat{\lambda}(z) = \frac{I_o \alpha}{j\omega} \sin \alpha z$$

$$(b) \quad \hat{I}(z) = I_o e^{-\alpha|z|}$$

$$d\ell_{\text{eff}} = \int_{-d\ell/2}^0 e^{\alpha z} dz + \int_0^{d\ell/2} e^{-\alpha z} dz$$

$$= \frac{2}{\alpha} (1 - e^{-\alpha d\ell/2})$$

$$\hat{\lambda}(z) = \begin{cases} \frac{I_o \alpha}{j\omega} e^{-\alpha z} & z > 0 \\ \frac{-\alpha I_o}{j\omega} e^{\alpha z} & z < 0 \end{cases}$$

$$(c) \quad \hat{I}(z) = I_o \cosh \alpha z$$

$$d\ell_{\text{eff}} = \int_{-d\ell/2}^{d\ell/2} \cosh \alpha z dz = \frac{2}{\alpha} \sinh \frac{\alpha d\ell}{2}$$

$$\hat{\lambda}(z) = \frac{-\alpha I_o}{j\omega} \sinh \alpha z$$

$$5. \quad \langle \vec{S} \rangle = \frac{1}{2} \operatorname{Re} \vec{E} \times \vec{H}^*$$

$$= \frac{1}{2} \frac{|m|^2 k^4 \eta}{(4\pi)^2 r^2} \sin^2 \theta \vec{i}_r$$

$$\langle P \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \langle S_r \rangle r^2 \sin \theta d\theta d\phi$$

$$= \frac{1}{2} \frac{|m|^2 k^4 \eta 2\pi}{(4\pi)^2} \int_{\theta=0}^{\pi} \sin^3 \theta d\theta$$

$$= \frac{|m|^2 k^4 \eta}{12\pi} = |I|^2 \frac{(dS)^2 k^4 \eta}{12\pi}$$

$$\langle P \rangle = \frac{1}{2} |I|^2 R \rightarrow R = \frac{k^2 \eta (dS)^2}{6\pi}$$

$$6. \quad (a) \quad (\hat{p}_z)_{\text{ind}} = 4\pi \epsilon_o R^3 \hat{E}_o$$

$$(b) \quad (\hat{p}_z)_{\text{ind}} = \frac{4\pi \epsilon_o (\epsilon - \epsilon_o) R^3 \hat{E}_o}{2\epsilon_o + \epsilon}$$

$$(c) \quad \langle P \rangle = \frac{\omega^4 |\hat{p}_{\text{ind}}|^2 \eta}{12\pi c^2}$$



# RADIATION

7. (a)

$$\bar{H}_{\text{dipole}} = \frac{m}{4\pi r^3} [2\cos\theta \bar{i}_r + \sin\theta \bar{i}_\theta] \quad (\text{from Eq. (8) Section (5.5.1)})$$

$$\bar{H}_{\text{sphere}} = H_o \left\{ \left(1 - \frac{R^3}{r^3}\right) \cos\theta \bar{i}_r - \left(1 + \frac{R^3}{2r^3}\right) \sin\theta \bar{i}_\theta \right\} \quad (\text{from Eq. (15) Section (5.7.2ii)})$$

$\bar{H}_{\text{sphere}}$  is sum of uniform field plus equivalent induced magnetic dipole moment.

$$H_o R^3 = \frac{-2m_{\text{ind}}}{4\pi} \rightarrow m_{\text{ind}} = -2\pi H_o R^3$$

$$(b) \quad \hat{E} = \frac{\hat{m}_{\text{ind}} j k^3}{4\pi} n e^{-jkr} \sin\theta \left[ \frac{1}{(jkr)} + \frac{1}{(jkr)^2} \right] \bar{i}_\phi$$

$$\hat{H} = \frac{-j\hat{m}_{\text{ind}}}{4\pi} k^3 e^{-jkr} \left\{ \bar{i}_r 2\cos\theta \left( \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) + \bar{i}_\theta \sin\theta \left( \frac{1}{(jkr)} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right\}$$

$$(c) \quad \text{From Prob. 5, } \langle P \rangle = \frac{|\hat{m}_{\text{ind}}|^2 k^4 \eta}{12\pi}$$

$$\langle P \rangle \propto \omega^4$$

$$8. (a) \quad \bar{B} = \nabla \times \bar{A} = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (A_\phi \sin\theta) \bar{i}_r - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \bar{i}_\theta$$

$$\frac{dr}{r d\theta} = \frac{B_r}{B_\theta} = \frac{-1}{\sin\theta} \frac{\frac{\partial}{\partial \theta} (A_\phi \sin\theta)}{\frac{\partial}{\partial r} (r A_\phi)}$$

$$\frac{\partial}{\partial r} (r \sin\theta A_\phi) dr + \frac{\partial}{\partial \theta} (r \sin\theta A_\phi) d\theta = 0$$

$$r \sin\theta A_\phi = \text{constant}$$

$$(b) \quad A_\phi = \text{Re} \left[ \frac{\hat{A}_\phi}{e^{j\omega t}} \right] = \text{Re} \left[ \frac{\mu \hat{m}}{4\pi r^2} \sin\theta (1 + jkr) e^{j(\omega t - kr)} \right]$$

Assume  $\hat{m} = m$  real

$$A_\phi = \frac{\mu m}{4\pi} \frac{\sin\theta}{r^2} [\cos(\omega t - kr) - k r \sin(\omega t - kr)]$$

# RADIATION

$$\text{Magnetic field lines} \rightarrow \sin^2 \theta \left[ \frac{\cos(\omega t - kr)}{kr} - \sin(\omega t - kr) \right] = \text{constant}$$

## Section 9.3

$$9. \quad (a) \quad r_1 \approx r - a \cos \theta, \quad r_2 \approx r + a \cos \theta$$

$$\hat{E}_\theta \approx \frac{\hat{E}_0}{jkr} \sin \theta e^{-jkr} [e^{jk a \cos \theta} + e^{-j(k a \cos \theta - \chi)}]$$

$$\approx \frac{2\hat{E}_0}{jkr} \sin \theta e^{-j(kr - \frac{\chi}{2})} [\cos(k a \cos \theta - \frac{\chi}{2})]$$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

$$(b) \quad \langle S_r \rangle = \frac{1}{2} \frac{|\hat{E}_\theta|^2}{\eta} = \frac{2|\hat{E}_0|^2}{k^2 r^2 \eta} \sin^2 \theta \cos^2 (k a \cos \theta - \frac{\chi}{2})$$

$$(c) \quad \langle S_r \rangle = 0 \text{ when } k a \cos \theta - \frac{\chi}{2} = (2n + 1) \frac{\pi}{2}$$

$$\theta = \cos^{-1} \left[ \frac{(2n + 1) \frac{\pi}{2} + \frac{\chi}{2}}{ka} \right]$$

$$\langle S_r \rangle_{\max} \text{ when } k a \cos \theta - \frac{\chi}{2} = n\pi$$

$$\theta = \cos^{-1} \left[ \frac{n\pi + \frac{\chi}{2}}{ka} \right]$$

$$(d) \quad 2a = \frac{\lambda}{2} \rightarrow ka = \frac{\pi}{2}$$

$$\text{Broadside} \rightarrow \theta = 90^\circ \rightarrow \chi = \pm 2n\pi$$

$$\text{Endfire} \rightarrow \theta = 0 \rightarrow \langle S_r \rangle = 0 \quad \text{no endfire}$$

$$(e) \quad \hat{E}_\theta = \frac{\hat{E}_0}{jkr} \sin \theta e^{-jkr} \sum_{n=-N}^N e^{jn(k a \cos \theta - \chi_0)}$$

# RADIATION

$$= \frac{\hat{E}_o}{jkr} \sin\theta e^{-jkr} \left\{ \frac{\sin(N + \frac{1}{2})(k\cos\theta - \chi_o)}{\sin \frac{1}{2}(k\cos\theta - \chi_o)} \right\}$$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

$$\langle S_r \rangle = \frac{1}{2} \frac{|\hat{E}_\theta|^2}{\eta} = \frac{|\hat{E}_o|^2}{k^2 r^2 \eta} \sin^2 \theta \left\{ \frac{\sin(N + \frac{1}{2})(k\cos\theta - \chi_o)}{\sin \frac{1}{2}(k\cos\theta - \chi_o)} \right\}^2$$

$$\langle S_r \rangle = 0 \rightarrow (N + \frac{1}{2})(k\cos\theta - \chi_o) = n\pi$$

$$\theta = \cos^{-1} \frac{\frac{n\pi}{N + \frac{1}{2}} + \chi_o}{ka}$$

$$\langle S_r \rangle_{\max} \rightarrow (N + \frac{1}{2})(k\cos\theta - \chi_o) = (2n + 1) \frac{\pi}{2}$$

$$\theta = \cos^{-1} \left[ \frac{\frac{(2n + 1)(\pi/2)}{N + \frac{1}{2}} + \chi_o}{ka} \right]$$

10. (a)  $\hat{E}_\theta = \frac{\sin\theta e^{-jkr}}{jkr} [\hat{E}_1 e^{jk\cos\theta} + \hat{E}_2 + \hat{E}_3 e^{-jk\cos\theta}]$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

(b)  $\langle S_r \rangle = \frac{1}{2} \frac{|\hat{E}_\theta|^2}{\eta}$

(c) (i)  $\hat{I}_1 = \hat{I}_3 = I_o, \hat{I}_2 = 2I_o \rightarrow \hat{E}_1 = \hat{E}_3 \equiv \hat{E}_o, \hat{E}_2 = 2\hat{E}_o$

$$\hat{E}_\theta = \frac{2\hat{E}_o \sin\theta e^{-jkr}}{jkr} [\cos(k\cos\theta) + 1] = \frac{4\hat{E}_o}{jkr} \sin\theta e^{-jkr} \cos^2\left(\frac{k\cos\theta}{2}\right)$$

$$\langle S_r \rangle = 0 \rightarrow \theta = 0, 180^\circ; \theta = \cos^{-1} \left[ \frac{(2n + 1)\pi}{ka} \right]$$

$$\langle S_r \rangle_{\max} \rightarrow \theta = \cos^{-1} \left[ \frac{2n\pi}{ka} \right]$$

(ii)  $\hat{I}_1 = \hat{I}_3 = I_o, \hat{I}_2 = -2I_o, \hat{E}_2 = -2\hat{E}_o, \hat{E}_1 = \hat{E}_3 = \hat{E}_o$

# RADIATION

$$\hat{E}_\theta = \frac{2\hat{E}_o}{jkr} \sin\theta e^{-jkr} [\cos(ka\cos\theta) - 1] = \frac{-4\hat{E}_o \sin\theta e^{-jkr}}{jkr} \sin^2\left(\frac{ka\cos\theta}{2}\right)$$

$$\langle S_r \rangle = 0 \rightarrow \theta = 0, 180^\circ; \theta = \cos^{-1} \left[ \frac{2n\pi}{ka} \right]$$

$$\langle S_r \rangle_{\max} \rightarrow \theta = \cos^{-1} \left[ \frac{(2n+1)\pi}{ka} \right]$$

$$(iii) \quad \hat{I}_1 = -\hat{I}_3 = jI_o, \quad \hat{I}_2 = I_o \rightarrow \hat{E}_1 = -\hat{E}_3 = j\hat{E}_o, \quad \hat{E}_2 = \hat{E}_o$$

$$\begin{aligned} \hat{E}_\theta &= \frac{\hat{E}_o \sin\theta e^{-jkr}}{jkr} [e^{jkac\cos\theta} - e^{-jkac\cos\theta} + 2j] \\ &= \frac{2\hat{E}_o \sin\theta e^{-jkr}}{kr} [1 + \sin(ka\cos\theta)] \end{aligned}$$

$$\langle S_r \rangle = 0 \rightarrow \theta = 0, 180; \theta = \cos^{-1} \left[ \frac{(4n+3)\pi}{2ka} \right]$$

$$\langle S_r \rangle_{\max} \rightarrow \theta = \cos^{-1} \left[ \frac{(4n+1)\pi}{2ka} \right]$$

$$11. (a) \quad dI = K_o dx$$

$$d\hat{E}_\theta = \frac{E'_o \sin\theta e^{-jkr}}{jkr} e^{jkx\sin\theta\cos\phi} dx; \quad E'_o = \frac{-K_o d\ell k^2 \eta}{4\pi}$$

$$\hat{E}_\theta = \frac{E'_o \sin\theta e^{-jkr}}{jkr} \int_{-L/2}^{+L/2} e^{jkx\sin\theta\cos\phi} dx$$

$$= \frac{E'_o \sin\theta e^{-jkr}}{(jkr) jk \sin\theta \cos\phi} \left[ e^{\frac{jkL}{2} \sin\theta \cos\phi} - e^{-\frac{jkL}{2} \sin\theta \cos\phi} \right]$$

$$= \frac{2jE'_o e^{-jkr}}{-k^2 r \cos\phi} \left[ \sin\left(\frac{kL}{2} \sin\theta \cos\phi\right) \right]$$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

$$(b) \quad \langle S_r \rangle = 0 \quad \frac{kL}{2} \sin\theta \cos\phi = n\pi$$

$$\langle S_r \rangle_{\max} = \frac{kL}{2} \sin\theta \cos\phi = \frac{(2n+1)\pi}{2}$$

## Section 9.4

$$\begin{aligned} 12. \quad (a) \quad \hat{E}_\theta &= \eta \hat{H}_\phi = \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r} \left\{ \int_{-L/2}^0 \left(1 + \frac{2z}{L}\right) e^{jkz\cos\theta} dz + \int_0^{L/2} \left(1 - \frac{2z}{L}\right) e^{jkz\cos\theta} dz \right\} \\ &= \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r} \left\{ \frac{e^{jkz\cos\theta}}{jk\cos\theta} \left(1 + \frac{2}{jkL\cos\theta} (jkz\cos\theta - 1)\right) \right\}_0^{-L/2} \\ &\quad + \frac{e^{jkz\cos\theta}}{jk\cos\theta} \left(1 - \frac{2}{jkL\cos\theta} (jkz\cos\theta - 1)\right) \Big|_0^{L/2} \Big\} \\ &= \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r} \left\{ 1 - \frac{2}{jkL\cos\theta} - e^{\frac{-jkL}{2}\cos\theta} \left(1 + \frac{2}{jkL\cos\theta} \left(\frac{-jkL}{2}\cos\theta - 1\right)\right) \right. \\ &\quad \left. + e^{\frac{jkL}{2}\cos\theta} \left(1 - \frac{2}{jkL\cos\theta} \left(\frac{jkL}{2}\cos\theta - 1\right) - \left(1 + \frac{2}{jkL\cos\theta}\right)\right) \right\} \\ &= \frac{k\eta I_o \sin\theta e^{-jkr}}{4\pi r} \left\{ \frac{-4}{jkL\cos\theta} + \frac{2}{jkL\cos\theta} \left(e^{\frac{jkL}{2}\cos\theta} + e^{\frac{-jkL}{2}\cos\theta}\right) \right\} \\ &= \frac{\eta I_o \sin\theta e^{-jkr}}{j\pi k r L \cos^2\theta} [-1 + \cos(\frac{kL}{2}\cos\theta)] \\ \langle S_r \rangle &= \frac{1}{2} \frac{|E_\theta|^2}{\eta} \end{aligned}$$

$$\begin{aligned} (b) \quad \hat{E}_\theta &= \eta \hat{H}_\phi = \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r} \int_{-L/2}^{L/2} \cos \frac{\pi z}{L} e^{jkz\cos\theta} dz \\ &= \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r} e^{jkz\cos\theta} \frac{[\frac{\pi}{L} \cos\theta \cos \frac{\pi z}{L} + \frac{\pi}{L} \sin \frac{\pi z}{L}]}{\left(\frac{\pi}{L}\right)^2 - (k\cos\theta)^2} \Big|_{z=-L/2}^{L/2} \\ &= \frac{jk\eta I_o \sin\theta e^{-jkr}}{4rL} \frac{1}{\left[\left(\frac{\pi}{L}\right)^2 - (k\cos\theta)^2\right]} \left(e^{\frac{jkL}{2}\cos\theta} + e^{\frac{-jkL}{2}\cos\theta}\right) \\ &= \frac{jkL\eta I_o \sin\theta e^{-jkr}}{8r} \frac{\cos[\frac{kL}{2}\cos\theta]}{\left[\left(\frac{kL}{2}\cos\theta\right)^2 - \left(\frac{\pi}{2}\right)^2\right]} \end{aligned}$$

# RADIATION

$$\langle S_r \rangle = \frac{1}{2} \frac{|E_\theta|^2}{\eta}$$

(c)  $kL \ll 1$

$$\begin{aligned} \text{(a)} \quad \hat{E}_\theta &\approx \frac{\eta I_o \sin\theta e^{-jkr}}{j\pi k r L \cos^2\theta} \frac{1}{2} \frac{k^2 L^2 \cos^2\theta}{4} \\ &= \frac{\eta I_o \sin\theta k L e^{-jkr}}{j8\pi r} \rightarrow (d\ell)_{\text{eff}} = \frac{L}{2} \end{aligned}$$

$$R = 40\pi^2 \left(\frac{L}{\lambda}\right)^2$$

$$\text{(b)} \quad \hat{E}_\theta \approx \frac{-jkL\eta I_o \sin\theta e^{-jkr}}{2\pi^2 r} \rightarrow (d\ell)_{\text{eff}} = \frac{2L}{\pi}$$

$$R = \frac{320L^2}{\lambda^2}$$